Prepotential Formulation of Lattice Gauge Theories

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TO
MY BEST FRIEND

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Chapter 1

Indroduction

In 1974, Wilson proposed the Lagrangian formulation of LGT [1] where the gauge field theory is discretized on the D-dimensional space-time lattice. In 1975, Kogut and Susskind derived a lattice Hamiltonian [2] in which only the $(d-1)$-dimensional space is discretized while the time variable remains continuous. Theoretically, the two formulations are equivalent in the continuum limit. However, they both involve spurious gauge degrees of freedom. In this thesis we study manifestly gauge invariant loop approach to lattice gauge theory (see chapter 3 and 4) by reformulating Kogut Sussking Hamiltonian [2] in terms of the prepotential operators. These prepotential operators are simply harmonic oscillators belonging to the fundamental representations of the gauge group.

As mentioned above, the main idea of the lattice approach to gauge theories [3–7] is to incorporate a non-perturbative cut-off in the theory. This lattice cut off is introduced by considering finite lattice spacing a. The quantum theory which describes the physical world is defined as the limit of a regulated theory with a short distance a (ultraviolet) cut off and a volume L (infrared) cutoff.

$$\text{Quantum Field Theory} = \lim_{a\to 0, L\to\infty} (\text{Lattice Theory})_{a,L} \tag{1.1}$$

Continuum theories are approached from the lattice theory by tuning a set of relevant parameters [8] to reach the limit where the lattice correlation length diverges. Note that, while working on lattice, the variables such as lattice mass (m_L) or lattice correlation lengths (ξ_L) are usually taken in lattice units and are dimensionless, i.e,

$$m_L = m_{\text{phys}} \times a \quad \Rightarrow \quad \xi_L \approx \frac{1}{m_L} \equiv \frac{1}{m_{\text{phys}} \times a}; \tag{1.2}$$

where, m_{phys} is the physical mass. It is clear from the above expression, that

- both m_L and ξ_L are dimensionless,
- since the physical mass m_{phys} is always finite, as the lattice spacing a approaches zero, the lattice correlation length must diverge.

Thus the continuum limit of a lattice gauge theory is reached only when the theory has diverging correlation lengths. In a statistical mechanical theory, the divergence of correlation length is a signature of second order phase transition. In renormalization group analysis of lattice gauge theory [8], the coupling acts as a parameter which determines the cutoff. Hence to get a continuum limit of a field theory defined with a lattice cutoff, one needs to find the points in the coupling parameter space where the corresponding statistical model reaches the critical point. In terms of the coupling in lattice gauge theories, the continuum limit is achieved when the bare coupling approaches zero [3]. To realize this let us consider the beta function, determining the cut-off dependence of the coupling, defined as,

$$\beta(g_0) = a\frac{d}{da}g_0(a), \tag{1.3}$$

where g_0 is the bare coupling of the theory and a is lattice spacing. Under renormalization group flow, the coupling and lattice spacing are not two independent parameter. The continuum limit is reached when g_0 approaches a fixed point g_F in the coupling parameter space. Hence it implies that $\beta(g_F) = 0$ in the continuum limit. The perturbative renormalization group analysis shows that at the strong coupling limit of the theory $\beta(g_F)$ does not tend to zero. Hence the strong coupling limit of the theory is not a fixed point and hence not the continuum limit. However, in the weak coupling limit of the theory, the perturbative beta function in 4-dimensions acquires the form:

$$\beta(g_0) = \beta_0 g_0^3 + \beta_1 g_0^5 + \mathcal{O}(g_0^7) \tag{1.4}$$

which clearly vanishes at vanishing coupling. The one loop and two loop contribution to beta function for SU(N) gauge theory with N_f fermionic species is [3]:

$$\begin{aligned}
\beta_0 &= \frac{1}{16\pi^2}\left(\frac{11N}{3} - \frac{2N_f}{3}\right) \\
\beta_1 &= \left(\frac{1}{16\pi^2}\right)^2\left(\frac{34N^2}{3} - \frac{10NN_f}{3} - \frac{N_f(N^2-1)}{N}\right)
\end{aligned} \tag{1.5}$$

Now, integrating beta function, one obtains the coupling as:

$$\frac{1}{g_0^2} = \beta_0 \log\left(\frac{1}{a^2\Lambda_0^2}\right) + \frac{\beta_1}{\beta_0}\log\left(\log\left(\frac{1}{a^2\Lambda_0^2}\right) + \mathcal{O}(g_0^2)\right), \tag{1.6}$$

or equivalently,

$$a = \Lambda_0^{-1}(g_0^2\beta_0)^{\frac{\beta_1}{2\beta_0^2}} \exp\left(-\frac{1}{2\beta_0 g_0^2}\right)(1 + \mathcal{O}(g_0^2)). \tag{1.7}$$

where, Λ_0 is the integration constant. Now, clearly we get that at weak coupling limit $\xi_L(\approx \frac{1}{m_{phys}\times a})$ diverges as expected in continuum limit.

We start with the introduction to Hamiltonian formulation in the next section.

1.1 The Hamiltonian approach

The Hamiltonian formulation of lattice gauge theory is intuitively appealing as one directly deals with the construction and study of the physical gauge invariant Hilbert space in terms of the fundamental operators of the theory. In the Hamiltonian approach one discretizes the spatial lattice and leaves the time variable continuous. This approach was first taken by Kogut and Susskind in [2]. Till date the major trends in the researches in the field of lattice gauge theories have mostly followed the Euclidean approach. This is because of the fact that Euclidean path-integral formalism is easily implementable for numerical simulations. In this formalism Monte-Carlo studies of lattice gauge theories yield numbers to be directly checked with particle data. The drawback of the Monte Carlo technique is that it is entirely numerical and one has no way of getting a feeling for what is essential and what is superfluous. On the other hand, the Hamiltonian approach, in principle, allows us to directly compute the physical spectrum non-perturbatively by diagonalizing the lattice Hamiltonian in the gauge invariant Hilbert space. Therefore, the Hamiltonian approach enables us to compute the energy levels and explicitly construct the corresponding physical states in terms of the basic canonical operators of the theory. However, the main problem with the Hamiltonian approach is that the above diagonalization procedure even for pure gauge theory and on a finite lattice involves a severe truncation of the infinite dimensional gauge invariant physical Hilbert space to some finite dimension. These truncations are done under various approximation schemes. The simplest and the oldest approximation scheme is the strong coupling expansion [2, 6, 9]. In the strong coupling limit ($g^2 \rightarrow \infty$) all loop states are eigenstates of the free Hamiltonian. The eigenvalues or the energies of these loop states, in the units of g^2, are directly proportional to their lengths and fluxes they carry. Therefore the strong coupling expansion allows us to compute the low energy spectrum by truncating the infinite dimensional physical Hilbert space to a finite dimensional Hilbert space spanned by loop states of small lengths carrying small fluxes. However, these results are completely unphysical because the continuum limit of lattice gauge theory lies at the other extreme weak coupling ($g^2 \rightarrow 0$) end (1.7). The other popular variational [10] methods in the Hamiltonian lattice gauge theory involve trial ground state wavefunctions and hence again sample a very small part of the full gauge invariant Hilbert space. The Hamiltonian approach has also been exploited to develop other non- perturbative methods such as t-expansions [11], plaquette expansion [12] and coupled cluster method [13, 14]. Monte Carlo techniques have also been developed to study the spectrum of the lattice gauge theory Hamiltonian [15]. There are also renormalization group improved approaches where the original Kogut-Susskind Hamiltonian [16] is modified by including distant lattice sites/links interactions in order to minimize the discretization error and to get closer to the continuum limit. Again all these methods, though non-perturbative, chop off the gauge invariant Hilbert space as in the case of strong coupling expansion. Therefore, it is an important problem to compute and characterize all possible gauge invariant states which are mutually independent. In this thesis we define SU(2) prepotential operators (see chapter 2) which enable us to explicitly construct an orthonormal and complete gauge invariant loop basis (see chapter 3). We also compute the matrix elements of the Hamiltonian in this basis (chapter 4). We generalize the ideas to SU(N) in chapter 5 and 6.

1.2 Loop Formulation

Like any continuum gauge theory, LGT also suffers hugely from irrelevant gauge degrees of freedom. Hence it is always desirable to remove these irrelevant or unphysical degrees of freedom from the theory. Note that, within the Hamiltonian framework, where one is interested in the Hilbert space of the theory, the gauge redundancy increases its dimension considerably. On lattice, each link of the lattice carries a SU(N) link operator. However, there exists SU(N) gauge invariance at each lattice site. Hence, the actual dimension of the physical Hilbert space in SU(N) gauge theory defined on lattice, is the dimension of the quotient space $\otimes_{links} SU(N)/\otimes_{sites} SU(N)$. Considering a d dimensional lattice with periodic boundary condition the dimension of physical Hilbert space at each lattice site is exactly $(N^2-1)(d-1)$, where N denotes the SU(N) group. We again address this issue quantitatively in section 3.1. Now, there are two ways to proceed with the Hamiltonian formulation like in any gauge theory. Either one can make a suitable gauge choice (gauge fixing) to cut down the gauge degrees of freedom or work with only gauge invariant or physical degrees of freedom. In pure gauge theries, the physical or gauge invariant variables are the Wilson loops. These Wilson loops are basically the parallel transport or holonomy of gauge field around a closed path in space. Infact, in the simplest case of electrodynamics this notion of holonomies existed in terms of electric fields or flux lines since the time of Faraday. In the non-abelian case, in order to get gauge invariant variables, holonomies are taken around closed paths and finally the trace is taken.

In lattice gauge theories, these gauge invariant Wilson loop operators, constructed out of holonomies, acting on the strong coupling vacuum creates the loop states which span the physical Hilbert space of the theory. Infact, lattice formulation is tailor made for loop formulation of gauge theories. This is because in lattice formulation of gauge theories the basic variables are the link operators or equivalently holonomies and not the gluon fields like in the continuum. The lattice cut-off also provides a natural cut-off for the loop states carrying discrete "electric fluxes". Infact, the most convenient description of loop formulation of lattice gauge theories is in terms of dual and electric field quantum numbers (see eqn. (3.13) and (3.28)). This dual formulation of lattice gauge theories has been a subject of study for very long time [17–19]. Kolawa [18], Bruggmann [20], Gambini [21, 22] and many others [13, 14, 23, 24] have used the truncated dual basis to study the spectrum of SU(2) and higher SU(N) lattice gauge theories.

The loop studies of Yang Mills theory also inspired people working in gravity to utilize loop approach. This was made possible by Ashtekars introduction of gauge theory like variables for canonical gravity, namely, a connection and its canonically conjugate "electric field" with internal SU(2) degrees of freedom [25]. This approach to gauge theory eventually lead to a new non- perturbative quantization of canonical gravity [26] known as loop quantum gravity. Again the dual approach to loop quantum gravity is the spin-foam approach, where the Hilbert space of loop quantum gravity theories consists of polymer-like excitations supported on graphs (spin network states) [27]. This spin foam approach is again very much close to lattice gauge theory as the lattice is replaced by arbitrary graphs. This again has renewed interests of reformulating lattice gauge theories in terms of loops [23, 28–30] in recent years.

1.3 The Mandelstam Constraints

The most serious problem associated with the loop formulation of gauge theories is that set of all possible loop states form a highly over-complete basis in the gauge invariant Hilbert space. This over-completeness is simply because all loop states are not linearly independent. This overcompleteness of the full loop Hilbert space implies constraints which are known as Mandelstam constraints [31]. They reflect the structure of the gauge group in the form of a set of relations between the loop states of the theory. More precisely, the Mandelstam constraints allow us to express products of Wilson loops in terms of the sum of the products of smaller number of loops implying that all the loops in the theory are not mutually independent (see chapter 3 for quantitative discussion). These identities were first introduced by Mandelstam for the gauge group O(3) [31]. Extension to GL(N) was achieved by Giles [32]. These identities have also been used for specific choices of gauge groups for researchers in the field of Loop Quantum Gravity [21, 28].

The Mandelstam constraints have been difficult to solve as they involve arbitrarily large number of non-local loop states of all shapes and sizes (see chapter 3). On the other hand, the solutions of the the Mandelstam constraints are of significance not only for writing the non-abelian gauge theories without any spurious loop degrees of freedom (see chapter 4) but also for computing the Hamiltonian spectrun in the weak coupling limit. This is because unlike strong couling limit, near the weak coupling or continuum ($g^2 \rightarrow 0$) limit loop states of arbitrary large sizes and fluxes will become relevant [18]. In fact, if we are only interested in the strong coupling limit of lattice gauge theory, this problem of over-completeness and the problem of solving the Mandelstam constraints is simple.

This is because for the low energy spectrum the loop states which are large and/or carry large fluxes can be safely ignored in $g_0 \to \infty$ limit. As a result, the relevant loop Hilbert space in the strong coupling limit consists of very few states and is of finite dimension. The Mandelstam constraints can be easily solved by constructing an orthonormal basis using Gram-Schmidt procedure as the number of loop states are finite. This is the reason why the Mandelstam constraints are not even mentioned in the huge amount of literature dealing with the strong coupling expansion. However, in the continuum $g_0 \to 0$ limit, the number of loops contributing to the spectrum grows and solving the Mandelstam constraints become an important problem. Infact, as stated by Gambini and Pullin in [21], (Chapter 12, pp. 303304): "The proliferation of loops when one considers larger lattices and higher dimensions completely washes out the advantages provided by the (loop) formalism".

In this thesis we define prepotential formulation of pure lattice gauge theories and show that all Mandelstam constraints can be locally solved using the prepotential operator. Further, we systematically develop ideas and techniques to reformulate lattice gauge theories in loop space without any spurious loop degrees of freedom.

1.4 Prepotential Formulation

The prepotential operators are the Schwinger bosons or harmonic oscillator n-plets in the fundamental represenations of the SU(N) gauge group [29, 30, 33–36]. The SU(N) electric fields, SU(N) link operators are explicitly constructed in terms of these prepotential operators. This prepotential formalism is shown to be invariant under $SU(N) \times U(1)^{N-1}$ gauge group. As the prepotential operators belong to the fundamental representations of the gauge group they transform exactly like matter fields. Thus in the prepotential formalism gauge and matter sectors of the theory are treated on the same footing. Further, the simple transformation property of prepotentials enables us to solve the non-abelian Gauss law locally at each lattice site. Further, using some ideas from group representation theory we write down all possible mutually orthonormal local solutions of the Gauss law. As these solutions are mutually independent they also solve the Mandelstam constraints. Having solved all the constraints, we compute the loop dynamics within the orthonormal loop basis locally site by site. The final results at different lattice site are then glued together through the Abelian gauge invariance.

To illustrate the scheme more clearly, we give flow-chart in figure 1.1.

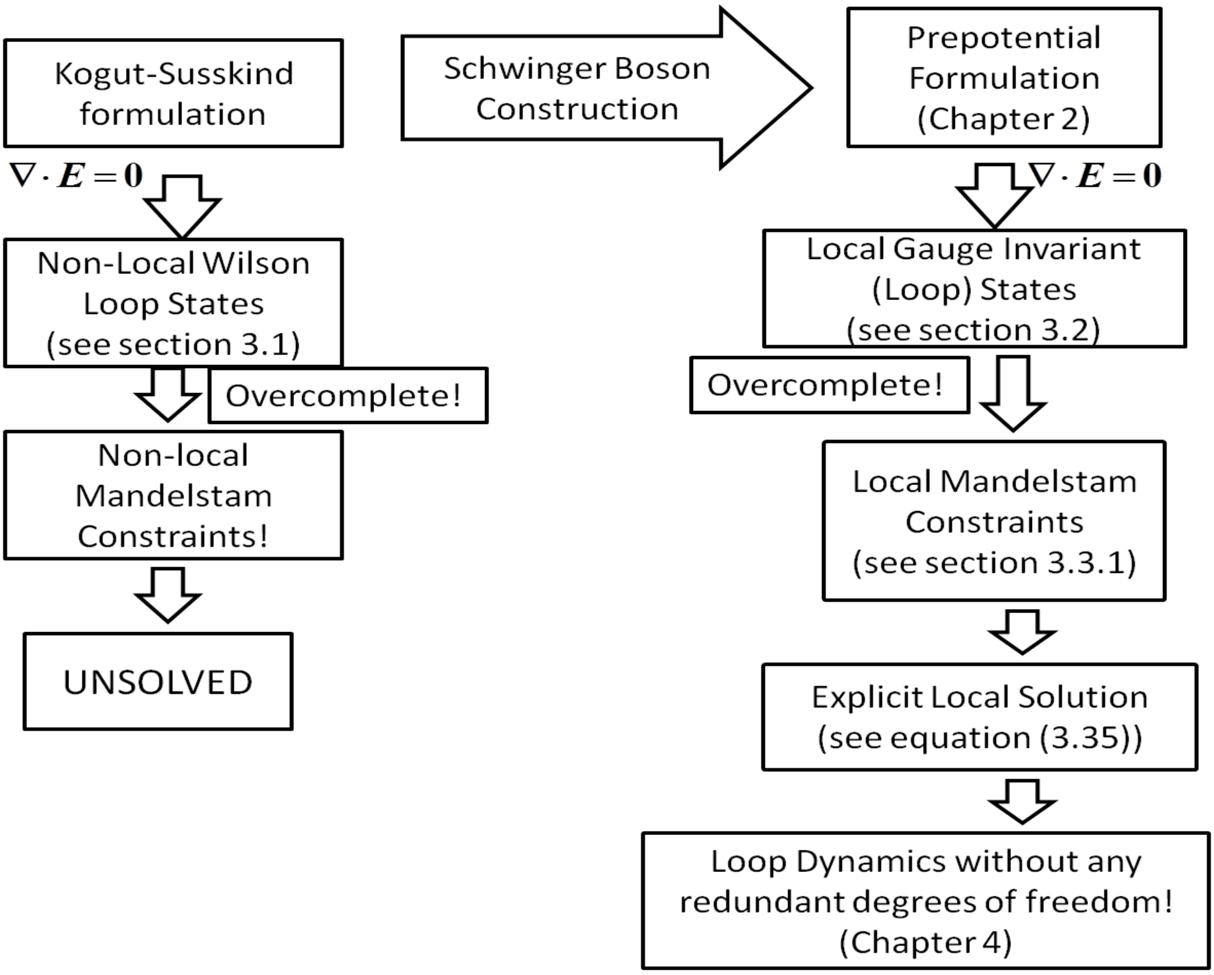

Figure 1.1: The Prepotential Hamiltonian Formulation

Overview of the thesis

The outline of the thesis as as follows. We start with the Hamiltonian formulation of lattice gauge theories in chapter 2. In this chapter we give a brief review and fix the notations for the thesis of Kogut-Susskind formulation of Hamiltonian lattice gauge theory. We discuss the Gauss law constraint as well as the electric field constraint. The next part of this chapter contains a new and equivalent formulation of Hamiltonian lattice gauge theories in terms of Prepotentials [29]. We define and construct the prepotential operators, construct the Kogut-Sussskind variables in terms of these. Rewrite the constraints and see their implications.

In chapter 3 we discuss the loop approach to lattice gauge theories. The gauge invariant operators in gauge theory are the Wilson loop operators. These Wilson loops acting on the strong coupling vacuum creates the states in the physical Hilbert space of the gauge theory. These states are highly non local due to arbitrary shapes and sizes of the loops. Moreover they form a much bigger set of basis vectors than the required dimension of physical Hilbert space. This is because of the fact that these Wilson loops are mutually dependent by a set of Mandelstam constraint. The present available literature does not contain a complete discussions on Mandelstam constrains. In this chapter we review the Mandelstam constraints discussed by Migdal in [37] so that we can compare this with the prepotential approach to loop formulation. We find that in terms of prepotentials [29] it is possible to write all the operators invariant under non-abelian gauge transformations locally at each site of the lattice. Being local they form a finite dimensional but overcomplete basis at each site. It is not at all difficult to relate these local loop operators in terms of the Mandelstam constraint between them. This local form of Mandelstam constraint is solvable leading to the exact orthonormal basis of the physical Hilbert space of SU(2) lattice gauge theory. Next we consider the dynamical issues in chapter 4. In this basis the dynamics is goverened by $3nj$ Wigner symbols.

In the next section, i.e in chapter 5 we generalize the prepotential formulation to gauge group SU(3). As stated earlier, the generalization from SU(2) to SU(3) is not at all trivial as the SU(3) group representation theoretic complications arise for SU(3). The origin of such complications is the multiplicity problem. We first solve these multiplicity problem by defining and constructing a new set of Schwinger bosons or irreducible schwinger bosons which makes SU(3) representations as simple as SU(2). We exploit this set of irreducible Schwinger bosons to contruct SU(3) irreducible prepotentials. In terms of the SU(3) irreducible prepotentials the SU(3) gauge invariant loop states can be constructed locally. We have also obtained all the Mandelstam constrain in their local form. Furthermore the SU(3) irreducible Schwinger boson construction has more applications in the context of SU(3) representations. We also briefly discuss two such applications namely in the construction of SU(3) coherent states and in the calculation of SU(3) Clebsch-Gordon coefficients in the appendices.

In chapter 6 we further generalize the prepotential formulation for arbitrary SU(N) group and reproduce all the results previously obtained for SU(3) in particular. Chapter 7 of this thesis contains explicit calculation of the spectrum of lattice gauge theory for a smaller lattice consisting of only four sites. Finally we discuss the future scope of the formulation in eighth and the final chapter.

Chapter 2

Hamiltonian Formulation of Lattice Gauge Theories and Prepotentials

In this chapter we discuss the reformulation of the Kogut-Susskind formulation of Hamiltonian Lattice Gauge Theory in terms of prepotential operators [29, 30]. For the sake of simplicity we start with the prepotential formulation of SU(2) lattice gauge theory which is the simplest non abelian gauge group. The first part of this chapter consists of the notations and definitions regarding the Kogut-Susskind Hamiltonian formulation for SU(N) lattice gauge theory that we follow in the course of this thesis. We set our notation in terms of the conventional Kogut-Susskind variables, color electric field and link operators. We also describe the Gauss law constraints. In the later part we define and construct the prepotential operators through the non-abelian eleectric fields. We then compute the SU(2) link operators in terms of the prepotentials. We also study the gauge invariance of the theory written in terms of prepotentials.

2.1 Kogut-Susskind Hamiltonian formulation

In this particular formulation the space is taken to be discretized whereas the time remains continuous. The Hamiltonian of $SU(N)$ lattice gauge theory is [2]:

$$H = \sum_{n,i} \sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}(n,i)E^{\mathrm{a}}(n,i) + K \sum_{\Box} Tr\left(U_{\Box} + U_{\Box}^{\dagger}\right) \tag{2.1}$$

with,

$$U_{\Box} = U(n,i)U(n+i,j)U^{\dagger}(n+j,i)U^{\dagger}(n,j),$$

where K is the coupling constant, a$(= 1, 2, \cdots, (N^2-1))$ is the color index. For a d dimensional square lattice, each link (n, i), originating from n^{th} site in the i^{th} direction carries a link operator $U(n, i)$ which connects the left and right electric fields $E_L^{\mathrm{a}}(n, i), E_R^{\mathrm{a}}(n+i, i)$ situated at both the ends, i.e n^{th} and $(n+i)^{th}$ point on the lattice. The link operator is basically a SU(N) symmetric top, whose configuration (i.e the rotation matrix from space fixed to body fixed frame) is given by the operator valued $(N \times N)$ SU(N) matrix $U(n, i)$. The quantization rules [2] of this system is as follows:

$$\begin{aligned} [E_L^{\mathrm{a}}(n,i), U^{\alpha}{}_{\beta}(n,i)] &= -\left(T^{\mathrm{a}}U(n,i)\right)^{\alpha}{}_{\beta}, \\ \left[E_R^{\mathrm{a}}(n+i,i), U^{\alpha}_{\beta}(n,i)\right] &= \left(U(n,i)T^{\mathrm{a}}\right)^{\alpha}{}_{\beta}. \end{aligned} \tag{2.2}$$

where, T^{a} are the SU(N) generators in the fundamental representation satisfying $[T^{\mathrm{a}}, T^{\mathrm{b}}] = if^{\mathrm{abc}}T_{\mathrm{c}}$, where, f^{abc} are the SU(N) structure constants. $E_L(n, i)$ and $E_R(n + i, i)$ being the generators of left and the right gauge transformations are not independent. The right generators $E_R^{\mathrm{a}}(n+i, i)$ are the parallel transport of the left generator $E_L^{\mathrm{a}}(n, i)$ on the link (n, i):

$$E_R(n+i,i) = -U^{\dagger}(n,i)E_L(n,i)U(n,i). \tag{2.3}$$

In (2.3), $E_R(n+i,i) \equiv \sum_{\mathrm{a}} E_R^{\mathrm{a}}(n+i,i)T^{\mathrm{a}}$ and $E_L(n,i) \equiv \sum_{\mathrm{a}} E_L^{\mathrm{a}}(n,i)T^{\mathrm{a}}$. The left and the right electric fields on every link, being the SU(N) generators, satisfy:

$$[E_L^{\mathrm{a}}(n,i), E_L^{\mathrm{b}}(n,i)] = if_{\mathrm{abc}}E_L^{\mathrm{c}}(n,i), \quad [E_R^{\mathrm{a}}(n,i), E_R^{\mathrm{b}}(n,i)] = if_{\mathrm{abc}}E_R^{\mathrm{c}}(n,i). \tag{2.4}$$

Further, using (2.3), it is easy to show that E_L^{a} and E_R^{a} commute amongst themselves:

$$\left[E_L^{\mathrm{a}}(n,i), E_R^{\mathrm{b}}(m,j)\right] = 0. \tag{2.5}$$

and therefore mutually independent. By construction (i.e (2.3)) on each link they always satisfy the constraints:

$$\sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}(n,i)E^{\mathrm{a}}(n,i) \equiv \sum_{\mathrm{a}=1}^{N^2-1} E_L^{\mathrm{a}}(n,i)E_L^{\mathrm{a}}(n,i) = \sum_{\mathrm{a}=1}^{N^2-1} E_R^{\mathrm{a}}(n+i,i)E_R^{\mathrm{a}}(n+i,i). \tag{2.6}$$

Note that, the Hamiltonian in (2.1) involves the squares of either left or the right electric fields. Under gauge transformation ($\Lambda(n)$ at site n) the link operator and left & right electric fields transform as:

$$
\begin{aligned}
U(n,i) &\to \Lambda(n)U(n,i)\Lambda^\dagger(n+i), \\
E_L(n,i) \to \Lambda(n)E_L(n,i)\Lambda^\dagger(n), &\quad E_R(n+i,i) \to \Lambda(n+i)E_R(n+i,i)\Lambda^\dagger(n+i).
\end{aligned} \tag{2.7}
$$

The Hamiltonian (2.1) and the basic commutation relations (2.2) are invariant under the SU(N) gauge transformations (2.7). From (2.7), the $SU(N)$ Gauss law constraint at every lattice site n is

$$
G^{\mathrm{a}}(n) = \sum_{i=1}^{d}\Big(E^{\mathrm{a}}_L(n,i) + E^{\mathrm{a}}_R(n,i)\Big) = 0, \forall n \text{ and for a} = 1, \dots N^2-1. \tag{2.8}
$$

It is convenient to define the left and right strong coupling vacuum state $|0\rangle_L$ and $|0\rangle_R$ on every link which are annihilated by their corresponding electric fields:

$$
E^{\mathrm{a}}_L(n,i)|0,(n,i)\rangle_L = 0, \qquad E^{\mathrm{a}}_R(n+i,i)|0,(n+i,i)\rangle_R = 0, \quad \forall \text{ links } (n,i). \tag{2.9}
$$

We will denote the vacuum state on a link by $|0\rangle \equiv |0,(n,i)\rangle_L \otimes |0,(n,i)\rangle_R \equiv |0\rangle_L \otimes |0\rangle_R$, suppressing all the link as well as L, R indices. The quantization rules (2.2) show that the link operators $U^{\alpha}{}_{\beta}(n,i)$ acting on the strong coupling vacuum (2.9) create SU(N) fluxes on the links. As an example, using (2.2):

$$
\begin{aligned}
E^2_L(n,i)\Big(U^{\alpha}{}_{\beta}|0\rangle\Big) &= E^2_R(n+i,i)\Big(U^{\alpha}{}_{\beta}|0\rangle\Big) = \sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}_L\left[E^{\mathrm{a}}_L, U^{\alpha}{}_{\beta}\right]|0\rangle \\
&= -\sum_{\mathrm{a}=1}^{N^2-1}\left[E^{\mathrm{a}}_L, (-T^{\mathrm{a}}U)^{\alpha}{}_{\beta}\right]|0\rangle = (T^{\mathrm{a}})^{\alpha}{}_{\gamma}(T^{\mathrm{a}})^{\gamma}{}_{\delta}U^{\delta}{}_{\beta}|0\rangle \\
&= \left(\frac{1}{2}\delta^{\alpha}{}_{\delta}\delta^{\gamma}_{\gamma} - \frac{1}{2N}\delta^{\alpha}{}_{\gamma}\delta^{\gamma}_{\delta}\right)U^{\delta}{}_{\beta}|0\rangle = \frac{1}{2N}\left(N^2-1\right)\Big(U^{\alpha}{}_{\beta}|0\rangle\Big).
\end{aligned} \tag{2.10}
$$

The higher SU(N) irreducible flux eigenstates of E^2_L or E^2_R on a link are constructed by considering the states $U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_1}{}_{\beta_1}|0\rangle$ and symmetrizing the α and therefore also β indices according to certain SU(N) Young tableau.

2.2 Prepotential Formulation

In this section we define SU(2) prepotential operators as an alternate variables of the theory and reformulate the Hamiltonian as well as the associated constraints in terms of these. We will generalize these ideas and techniques to higher rank SU(3) and SU(N)($N > 3$) groups later in chapter 5 and 6 respectively.

2.2.1 Definition and Construction

As discussed in previous section, two electric fields are associated with each link of the lattice. We define SU(2) prepotential operators $a^\dagger(n,i;L)$ and $a^\dagger(n,i;R)$ associated with left and right end of the link (n,i). Using the Schwinger boson construction [33] of the angular momentum algebra, the left and the right electric fields on a link (n,i) can be written as:

$$\begin{aligned} \text{Left electric fields:} \qquad E^{\rm a}_L(n,i) &\equiv a^\dagger(n,i;L)\frac{\sigma^{\rm a}}{2}a(n,i;L), \\ \text{Right electric fields:} \qquad E^{\rm a}_R(n+i,i) &\equiv a^\dagger(n+i,i;R)\frac{\sigma^{\rm a}}{2}a(n+i,i;R). \end{aligned} \tag{2.11}$$

In (2.11), $a_\alpha(n,i;l)$ and $a^\dagger_\alpha(n,i;l)$ are the doublets of harmonic oscillator creation and annihilation operators with $l = L, R, \alpha = 1, 2$ satisfying the following algebra:

$$\begin{aligned} \left[a_\alpha(n,i;l), a^\dagger_\beta(n',i';l')\right] &= \delta_{nn'}\delta_{ii'}\delta_{ll'}\delta_{\alpha\beta} \\ \left[a_\alpha(n,i;l), a_\beta(n',i';l')\right] &= \left[a^\dagger_\alpha(n,i;l), a^\dagger_\beta(n',i';l')\right] = 0 \end{aligned} \tag{2.12}$$

Like $E^{\rm a}_L(n,i)$ and $E^{\rm a}_R(n+i,i)$, the locations of $a(n,i,L), a^\dagger(n,i,L)$ and $a(n+i,i,R), a^\dagger(n+i,i,R)$ are on the left and the right of the link (n,i). For notational convenience we suppress the link indices and denote $a^\dagger(n,i,L)$ and $a^\dagger(n+i,i,R)$ by $a^\dagger(L)$ and $a^\dagger(R)$ respectively. This is clearly illustrated in Figure 2.1. Note that the relations (2.11) imply that the strong coupling vacuum (2.9) is the harmonic oscillator vacuum. Under SU(2) gauge transformation, the prepotential harmonic oscillators transform as SU(2) doublets:

$$\begin{aligned} a^\dagger_\alpha(L) \to a^\dagger_\beta(L)\ \left(\Lambda^\dagger_L\right)^\beta{}_\alpha, \qquad & a^\dagger_\alpha(R) \to a^\dagger_\beta(R)\ \left(\Lambda^\dagger_R\right)^\beta{}_\alpha \\ a^\alpha(L) \to \left(\Lambda_L\right)^\alpha{}_\beta\ a^\beta(L), \qquad & a^\alpha(R) \to \left(\Lambda_R\right)^\alpha{}_\beta\ a^\beta(R). \end{aligned} \tag{2.13}$$

One can also define $\tilde{a}^{\dagger\alpha} = \epsilon^{\alpha\beta}a^\dagger_\beta$ and $\tilde{a}_\alpha = \epsilon_{\alpha\beta}a^\beta$ which under $SU(2)$ transformation transform as a^α and $a^\dagger_\alpha$ respectively. Here, $\epsilon_{\alpha\beta}$ is a completely antisymmetric tensor ($\epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1$).

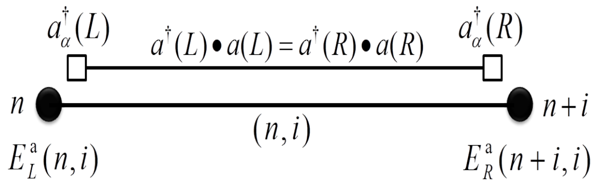

Figure 2.1: The left and right electric fields and the corresponding prepotentials in SU(2) lattice gauge theory. We have denoted $a^\dagger(n,i,L)$ and $a^\dagger(n+i,i,R)$ by $a^\dagger(L)$ and $a^\dagger(R)$ respectively. The unoriented abelian flux line connecting them represents the U(1) Gauss law (2.19) constraint.

In addition to the elecric fields, each of the links also carry a link variable $U^\alpha{}_\beta(n,i)$. For notational simplicity from now on we will remove the link index (n,i) while considering a single link at a time. The basic SU(2) flux states on each link are created by the action of link variables on the strong coupling vacuum. The flux state must carry a SU(2) representation and is characterized by the SU(2)quantum numbers, i.e $|j,m\rangle$. As discussed in previous section, the link variable transforms by both the gauge transformation seating at its both ends. In terms of link operators, $U^\alpha{}_\beta$ on a link l the basic SU(2) flux states on the link l can be constructed as:

$$|j_L(l),m_L(l)\rangle\otimes|j_R(l)m_R(l)\rangle = N\underbrace{\left(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}..U^{\alpha_{2j}}{}_{\beta_{2j}}+...\right)}_{\text{(2j)! permutations}}|0\rangle. \tag{2.14}$$

In (2.14), N is the normalization factor. $j_L(l)=j_R(l)\equiv j(l)$ because both the left and right states are created by action of the same $U^\alpha{}_\beta$ where, α is the left index and β is the right index. $m_L=\sum_{i=1}^{2j}\alpha_i$ and $m_R=\sum_{i=1}^{2j}\beta_i$ with $\alpha_i,\beta_i=\pm\frac{1}{2}$. The $(2j)!$ terms in (2.14) are required to implement the symmetries of SU(2) Young tableau in the left $(\alpha_1\alpha_2\cdots\alpha_{2j})$ as well as the right $(\beta_1\beta_2\cdots\beta_{2j})$ indices.

Now, in terms of prepotentials, which are defined at each end of the links; so we can create left and right states by the left and right prepotentials in the following way:

$$|j(l),m_L(l),m_R(l)\rangle = |j(l),m_L(l)\rangle_L\otimes|j(l),m_R(l)\rangle_R, \tag{2.15}$$

where,

$$\begin{aligned} |j(l), m_L(l)\rangle_L &= N_L a^\dagger_{\alpha_1}(L) a^\dagger_{\alpha_2}(L) \cdots a^\dagger_{\alpha_n}(L)|0\rangle_L \equiv \hat{\mathcal{L}}_{\alpha_1\alpha_2\cdots\alpha_n}|0\rangle_L, \\ |j(l), m_R(l)\rangle_R &= N_R a^\dagger_{\beta_1}(R) a^\dagger_{\beta_2}(R) \cdots a^\dagger_{\beta_n}(R)|0\rangle_R \equiv \hat{\mathcal{R}}_{\beta_1\beta_2\cdots\beta_n}|0\rangle_R. \end{aligned} \tag{2.16}$$

In (2.16), N_L, N_R are normalization factors, $n = 2j(l)$, $m_L = \sum_{i=1}^{2j} m_i$ and $m_R = \sum_{i=1}^{2j} \tilde{m}_i$ with $m_i = \frac{1}{2}(\delta_{\alpha_i,1} - \delta_{\alpha_i,2})$ and $\tilde{m}_i = \frac{1}{2}(\delta_{\beta_i,1} - \delta_{\beta_i,2})$. The operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ are the $SU(2)\otimes U(1)$ flux creation operators at the left and right end of every link and create states in the prepotential Hilbert space $\mathcal{H}_p^{SU(2)}$. Note that these states are $SU(2)$ irreducible as they are symmetric in all the $SU(2)$ spin half indices and are defined for later convenience.

The gauge theory Hilbert space $\mathcal{H}_g$ is spanned by direct product of states of type (2.14) on all the lattice links. Note that as the flux value $j \to \infty$ on various links[1], the construction of the gauge theory Hilbert space $\mathcal{H}_g$ through (2.14) becomes more and more tedious. But the same construction trivializes in terms of prepotentials through 2.16.

From (2.14) and (2.16) we conclude that the Hilbert space $\mathcal{H}_p$ created using the prepotential operators on all lattice links is also the SU(2) gauge theory Hilbert space:

$$\mathcal{H}_g^{SU(2)} \equiv \mathcal{H}_p^{SU(2)}. \tag{2.17}$$

However, the construction of $\mathcal{H}_g$ using the prepotentials (2.16) is much simpler than the equivalent construction (2.14) using the link operators. This simplicity occurs because unlike the link operators $U_{\alpha\beta}(n,i)$ which are associated with links, the prepotential operators are attached to the sites (i.e, left or right ends of every link). Further, all the SU(2) prepotential creation operators commute amongst themselves and we do not need $(2j)!$ terms (as in (2.14)) to get the symmetries of SU(2) Young tableau. In words, the symmetries of SU(2) Young tableau are inbuilt in SU(2) prepotential operators. In the case of SU(N) gauge theory with $N \geq 3$ [35, 36], the identification (2.17)is no longer valid. In fact $\mathcal{H}_g^{SU(N)} \subset \mathcal{H}_p^{SU(N)}$ because of the existence of certain SU(N) gauge invariant operators for $N \geq 3$. We will discuss this issue in detail in the chapter 5 and 6 while discussing prepotential formulation for SU(3) and SU(N) respectively.

[1] These large j configurations are expected to dominate in the continuum ($g \to 0$) limit.

2.2.2 The additional $U(1)$ gauge invariance

One can see it clearly that the defining equations (2.11) for the prepotential operators are invariant under $U(1) \otimes U(1)$ gauge transformations on every link:

$$a^\dagger_\alpha(L) \rightarrow e^{i\theta(L)} a^\dagger_\alpha(L), \qquad a^\dagger_\alpha(R) \rightarrow e^{-i\theta(R)} a^\dagger_\alpha(R). \tag{2.18}$$

Note that the above abelian gauge transformations are defined on the two sides of every link and are independent of the SU(2) gauge transformations (2.13) which are defined at every lattice site. Using (2.11), the electric field constraints (2.6) on the links become the number operator constraints in terms of the prepotential operators:

$$\hat{N}(L) \equiv a^\dagger(L) \cdot a(L) = \hat{N}(R) \equiv a^\dagger(R) \cdot a(R) \equiv \hat{N} \tag{2.19}$$

In (2.19), $\hat{N} \equiv \hat{N}(n,i)$ and imply $\theta(L) = \theta(R)$ on every link which reduces the extra $U(1) \otimes U(1)$ gauge invariance to U(1). This is also clear from the fact that each state on both side of the link is being created by the action of link operator. Hence the total flux of the states must be same. Actually in terms of prepotentials both the ends are decoupled except the U(1) gauge invariance. We will again discuss these issues in a great detail while discussing prepotentials for higher SU(N) gauge groups. We can summarize by stating that in the prepotential formulation non-abelian fluxes can be absorbed locally at a site and the abelian fluxes spread along the links. Both the gauge symmetries together lead to non-local (involving at least a plaquette) Wilson loop states.

2.2.3 Link operators

The equations (2.11) already defines the left and right electric fields in terms of the prepotentials. To establish complete equivalence, we now write down the link operators explicitly in terms of the prepotentials. From $SU(2)$ gauge transformations of the link operator in (2.7) and $SU(2) \otimes U(1)$ gauge transformations (2.13), (2.18) of the prepotentials,

$$U^\alpha{}_\beta = \tilde{a}^{\dagger\alpha}(L)\, \eta\, a^\dagger_\beta(R) + a^\alpha(L)\, \theta\, \tilde{a}_\beta(R), \tag{2.20}$$

where η and θ are functions of $SU(2)$ invariant number operator. The action of link operator in eqn. (2.20) on any SU(2) state residing on the link is graphically illustrated

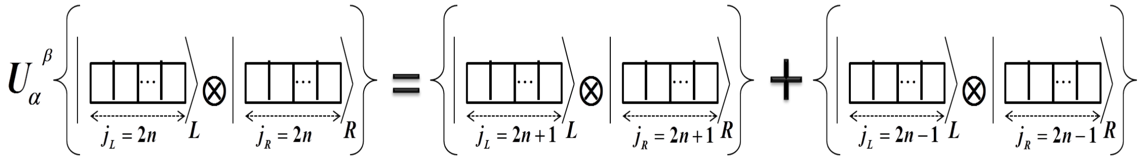

Figure 2.2: The Young tableau interpretation of the SU(2) link operator U in terms of the prepotential operators (2.20) acting on a state with $n_L = n_R = 2j$. The two terms in (2.20) correspond to the two sets of Young tableaus on the right hand side of this figure respectively.

in terms of SU(2) Young tableaus in Figure 2.2.

Since prepotentials are doublets, the link operator is a 2×2 matrix and must be SU(2) valued. Again as we have already mentioned the prepotentials decouple the left and right part of a link which are only connected by the number operator constraint discussed in the last section. This is obvious in the explicit matrix form of the link operator written as the product of the left part U_L and the right part U_R as:

$$U = \underbrace{\begin{pmatrix} a_2^\dagger(L)\eta_L & a_1(L)\theta_L \\ -a_1^\dagger(L)\eta_L & a_2(L)\theta_L \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} \eta_R a_1^\dagger(R) & \eta_R a_2^\dagger(R) \\ \theta_R a_2(R) & \theta_R(-a_1(R)) \end{pmatrix}}_{U_R} \tag{2.21}$$

Where, $\eta_L, \eta_R, \theta_L, \theta_R$ are the left and right invariants constructed out of number operators. ¿From (2.20) it follows that, $\eta = \eta_L \eta_R, \qquad \theta = \theta_L \theta_R$. Since U_L and U_R are themselves SU(2) matrix, they must satisfy unitarity by themselves. From (2.21):

$$U_L^\dagger U_L = \begin{pmatrix} \bar{\eta}_L \left(\hat{N}+2\right) \eta_L & 0 \\ 0 & \bar{\theta}_L \hat{N} \theta_L \end{pmatrix}, \quad U_R U_R^\dagger = \begin{pmatrix} \eta_R \hat{N} \bar{\eta}_R & 0 \\ 0 & \theta_R \left(\hat{N}+2\right) \bar{\theta}_R \end{pmatrix} \tag{2.22}$$

In (2.22), $\hat{N} = a^\dagger(L) \cdot a(L) = a^\dagger(R) \cdot a(R)$ is the common number operator (2.19) on the link (n, i). Therefore, for $U_{\alpha\beta}$ to be unitary as well as unimodular we get:

$$\eta_L = \frac{1}{\sqrt{\hat{N}+2}}, \quad \theta_L = \frac{1}{\sqrt{\hat{N}}}, \quad \eta_R = \frac{1}{\sqrt{\hat{N}}}, \quad \theta_R = -\frac{1}{\sqrt{\hat{N}+2}}. \tag{2.23}$$

Finally, the link operator can be disentangled into its left and right parts as:

$$U = \underbrace{\frac{1}{\sqrt{\hat{N}+1}} \begin{pmatrix} a_2^\dagger(L) & a_1(L) \\ -a_1^\dagger(L) & a_2(L) \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} a_1^\dagger(R) & a_2^\dagger(R) \\ a_2(R) & -a_1(R) \end{pmatrix} \frac{1}{\sqrt{\hat{N}+1}}}_{U_R} \equiv U_L \, U_R \tag{2.24}$$

and satisfies $U^\dagger U = UU^\dagger = 1$. Moreover using (2.12) one can show explicitly that,

$$[U_{\alpha\beta}, U_{\gamma\delta}] = \left[U_{\alpha\beta}, U^\dagger_{\gamma\delta}\right] = 0 \tag{2.25}$$

Note that, the above relation is non-trivial as $U_{\alpha\beta}$ involve both creation and annihilation operators.

2.3 Summary and Discussion

In this chapter we have redefined the conventional canonically conjugate variables of the lattice gauge theory Hamiltonian in terms of a set of new variables called prepotentials. We have re-expressed the color electric fields as well as the link variables in terms of prepotentials. The most important feature of this formulation is that the non-abelian gauge invariance of the theory gets confined to prepotentials located around lattice site with additional abelian gauge invariance involving prepotentials along every link. This is manifested in the construction of the link operator (2.24) which breaks up into U_L and U_R which transform as matter fields at the left and right ends of the link. It is also worth mentioning that all the states in the gauge theory Hilbert space of the theory can be constructed very easily in terms of prepotential as shown in (2.16) compared to the conventional construction in (2.14).

In the next chapter we exploit the prepotential formulation of SU(2) Hamiltonian lattice gauge theory to construct all possible mutually independent loop states.

Chapter 3

Prepotentials and Loop Formulation

In this chapter we discuss Mandelstam constraints in detail, first in terms of Kogut Susskind link operators and then in terms of prepotential operators. We show that the non-local Mandelstam constraints become local in the prepotential formulation.

3.1 Wilson loops and Mandelstam constraints on lattice

For a d-dimensional periodic lattice with n^d sites and dn^d links, the dimension of the physical Hilbert space ($\mathcal{N}$) is the dimension of the quotient space $\otimes_{links} SU(N)/\otimes_{sites} SU(N)$, as each link carries a SU(N) link operator while Gauss law is satisfied at each site.

$$\mathcal{N}^{SU(N)} = (N^2 - 1)(d - 1)n^d. \tag{3.1}$$

Hence there is only $(N^2 - 1)(d - 1)$ physical degrees of freedom locally at each site of the lattice. However, the loop Hilbert space or the space of all possible loop states is always of a bigger dimension.

In the last chapter we have defined the prepotential operators at both ends of a link. Now if we consider a particular site on a d dimensional lattice, it has $2d$ number of links connected to it. Each link must carry a set of prepotential doublet at that end. Hence, at a particular site, we have $2d$ number of prepotential doublet attached to each of the $2d$ links emerging from the site. The SU(2) invariants are anti symmetric combination of any two different prepotential doublets. In terms of Young tableaux it can be understood well as each prepotential operator represents a single Young tableaux box. The SU(2) invariant is a column of two boxes. Hence, for a site in 2d lattice, we have total $2d$ different prepotential doublet. The total number of invariants that we can construct is now ${}^{2d}C_2$. This is precisely the dimension of the loop Hilbert space of SU(2) gauge theory at each site. But from (3.1), the dimension of physical Hilbert space is $3(d-1)$ per lattice site. Note that, in terms of SU(2) prepotentials there exists an additional abelian gauge invariance as discussed in last chapter. This extra abelian gauge invariance will put a constraint in each direction. So, in d dimensional lattice, there exist d number of U(1) Gauss law constraint. Hence we are left with

$$\mathcal{M}^{SU(2)} = {}^{2d}C_2 - d = 2d(d-1)$$

loop degrees of freedom per lattice site. Now it is clear that, there should be additional

$$\mathcal{M}^{SU(2)} - \mathcal{N}^{SU(2)}/n^d = 2d(d-1) - 3(d-1) = 2d^2 - 5d + 3 \tag{3.2}$$

number of constraints on the local loop basis to get orthonormal loop basis at each site. Hence there should be exactly $2d^2 - 5d + 3$ number of Mandelstam constraints locally at each site. Note that for $d = 2$ we have only one Mandelstam constraint per site and $d = 3$ we have six Mandelstam constraints per site.

We immediately see that this precise counting of Mandelstam constraints present in the theory can be obtained in prepotential approach. We now review them briefly in terms of the standard link operator language. This will also highlight why the Mandelstam constraints have been so notoriously difficult to solve. To illustrate the overcompleteness of non-local Wilson loops as well as associated Mandelstam constraints on lattice, let us first consider the simple example of SU(2) gauge theory for a 2-d lattice consisting of only two plaquettes. The Wilson loops existing for such a system is given in Figure (3.1a,b,c) respectively. Although the lattice size as well as dimension of space is small even then the loop states form an overcomplete basis. There exist only three possible loops for that system as given in (3.1a,b,c). These three Wilson loop operators satisfy:

$$(\mathrm{Tr}W_A)\,(\mathrm{Tr}W_B) \equiv \mathrm{Tr}\,(W_A W_B) + \mathrm{Tr}\left(W_A W_B^{-1}\right). \tag{3.3}$$

The Mandelstam identity or constraint follows from the fact that holonomies are nothing but general SU(2) matrices and hence can always be written in the form $W = W^0\mathbb{I} + iW^i\sigma^i$,

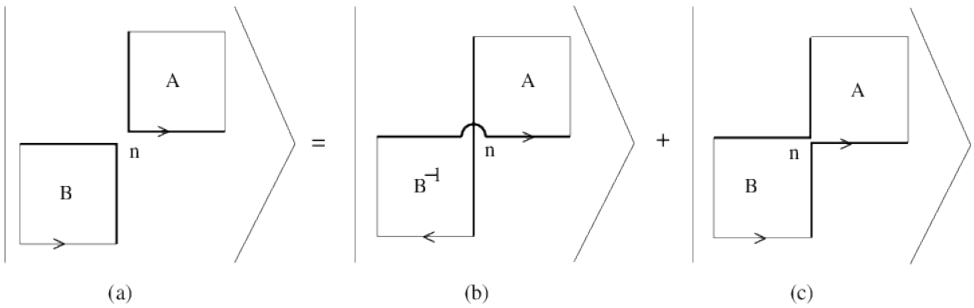

Figure 3.1: Mandelstam constraint for two plaquette lattice. The solid lines denotes intertwining between two plaquettes.

where, σ^i's are the Pauli matrices. Note that the operator relation given in (3.3) is a well-known identity for any two SU(2) matrices W_A and W_B with the most general form given by $W_X = X_0 1 + i\sum_{a=1}^{3} X_a\sigma^a$ where σ^a are the Pauli matrices, X_0, X_a are real and satisfy $X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1$. As Pauli matrices are traceless, (3.3) comes directly as:

$$2{W_A}^0 2{W_B}^0 = 2{W_A}^0{W_B}^0 - {W_A}^i{W_B}^i + 2{W_A}^0{W_B}^0 + {W_A}^i{W_B}^i. \tag{3.4}$$

Now we can associate Wilson loop states with each of these three loop operators as follows:

$$|\gamma_1\rangle \equiv (\mathrm{Tr}W_A)\,(\mathrm{Tr}W_B)\,|0\rangle, \qquad |\gamma_2\rangle \equiv \mathrm{Tr}\left(W_A W_B^{-1}\right)|0\rangle, \qquad |\gamma_3\rangle \equiv \mathrm{Tr}\,(W_A W_B)\,|0\rangle, \tag{3.5}$$

The identity (3.3) implies the fundamental Mandelstam constraint in 2d:

$$|\gamma_1\rangle = |\gamma_2\rangle + |\gamma_3\rangle. \tag{3.6}$$

Thus we see that the three loop states $|\gamma_1\rangle, |\gamma_2\rangle$ and $|\gamma_3\rangle$ are linearly dependent.

Note that the above discussed simplest Mandelstam constraint is there for loops carrying only one unit of flux ($j = 1/2$). As soon as it starts carrying more and more fluxes, life becomes complicated even for this small lattice. Only two plaquettes but each carrying two units of fluxes ($j = 1$) case implies seven mutually dependent states as demonstrated in 3.2. To appreciate the problem, let us consider the most general loop states constructed

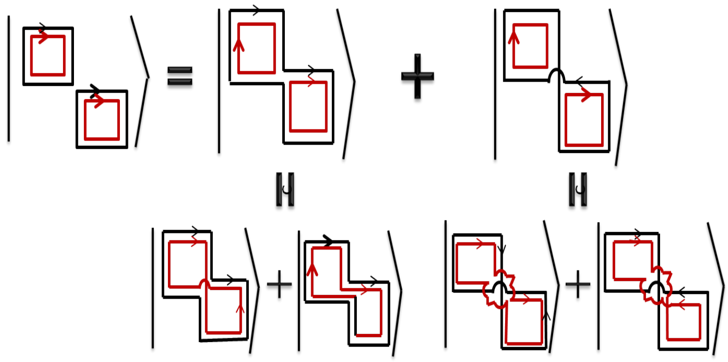

Figure 3.2: Loops involving two units of fluxes around a two plaquette lattice.

for these two plaquettes A and B:

$$
\begin{aligned}
|N_A, N_B\rangle &\equiv (\mathrm{Tr}W_A)^{N_A}(\mathrm{Tr}W_B)^{N_B}|0\rangle \\
&= (\mathrm{Tr}W_A)^{N_A-1}(\mathrm{Tr}W_B)^{N_B-1}(\mathrm{Tr}W_AW_B + \mathrm{Tr}W_AW_B^{-1})|0\rangle \\
&= (\mathrm{Tr}W_A)^{N_A-2}(\mathrm{Tr}W_B)^{N_B-2}(\mathrm{Tr}W_AW_B + \mathrm{Tr}W_AW_B^{-1})^2|0\rangle \\
&. \\
&. \\
&= (\mathrm{Tr}W_A)^{N_A-N_{min}}(\mathrm{Tr}W_B)^{N_B-N_{min}}(\mathrm{Tr}W_AW_B + \mathrm{Tr}W_AW_B^{-1})^{N_{min}}|0\rangle \quad (3.7)
\end{aligned}
$$

where N_A, N_B are two arbitrary integers representing the SU(2) fluxes over A and B and $N_{\rm min}$=Minimum(N_A, N_B). The simple example, where $N_A = 2$ and $N_B = 2$, all possible loop states have been illustrated in figure(3.2). Thus, for the two plaquette sytem carryning $|N_A, N_B\rangle$ units of flux, there exist very large number of distinct but linearly dependent Wilson loop states contained in the relations (3.7). Note that for weak coupling limit it is necessary to include loops carrying all possible fluxes and that too for a arbitrary large lattice! To appreciate the difficulty we can further extend this simple two plaquette system with addition of one more plaquette. Before that, it is worth mentioning that, in general, if more than two loops in SU(2) lattice gauge theory passes through a particular site, there exists a set of identities satisfied by these Wilson loops as discussed in [37]. For the set of r ($r > 2$) loops $\Gamma_1(n)$, $\Gamma_2(n), \cdots, \Gamma_r(n)$ all based at lattice site n. These loops start from n in the direction $i_1, i_2, \cdots i_r$ and come back to n from directions

$j_1, j_2, \cdots j_r$ respectively. Then the products of these Wilson loops satisfies:

$$\sum_{\substack{\alpha_{i_1}\cdots\alpha_{i_r}\\ \beta_{j_1}\cdots\beta_{j_r}}} \epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}} \epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} \left(W(\Gamma_1(n))^{\alpha_{j_1}}{}_{\beta_{i_1}} \left(W(\Gamma_2(n))\right)^{\alpha_{j_2}}{}_{\beta_{i_2}} \cdots \left(W(\Gamma_r(n))\right)^{\alpha_{j_r}}{}_{\beta_{i_r}} \equiv 0. \quad (3.8)$$

Using the identities

$$\epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}} \epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} = \delta^{\beta_{j_1}}_{\alpha_{i_1}} \delta^{\beta_{j_2}}_{\alpha_{i_2}} \cdots \delta^{\beta_{j_r}}_{\alpha_{i_r}} - \delta^{\beta_{j_1}}_{\alpha_{i_2}} \delta^{\beta_{j_2}}_{\alpha_{i_i}} \cdots \delta^{\beta_{j_r}}_{\alpha_{i_r}} + \cdots$$

(3.8) can be written in terms of traces of Wilson loops [37]:

$$\begin{aligned} & \mathrm{Tr}W(\Gamma_1)\mathrm{Tr}W(\Gamma_2)\cdots\mathrm{Tr}W(\Gamma_r) \\ - \; & \mathrm{Tr}W(\Gamma_1\Gamma_2)\mathrm{Tr}W(\Gamma_3)\mathrm{Tr}W(\Gamma_4)\cdots\mathrm{Tr}W(\Gamma_r) + \cdots = 0. \end{aligned} \quad (3.9)$$

This general identity can be illustrated for 3 loops W_1, W_2, W_3 (may be three plaquettes only) in SU(2) gauge theory passes through a particular lattice site. This example is as well an extension of the earlier two plaquette scenario to the same involving 3 plaquettes. These three loops satisfy the relation:

$$\begin{aligned} & \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2,\beta_3}} \epsilon_{\alpha_1\alpha_2\alpha_3} \epsilon^{\beta_1\beta_2\beta_3} W_1^{\alpha_1}{}_{\beta_1} W_2^{\alpha_2}{}_{\beta_2} W_3^{\alpha_3}{}_{\beta_3} \\ = \; & \Big[\delta_{\alpha_1}{}^{\beta_1} \delta_{\alpha_2}{}^{\beta_2} \delta_{\alpha_3}{}^{\beta_3} - \delta_{\alpha_1}{}^{\beta_2} \delta_{\alpha_1}{}^{\beta_2} \delta_{\alpha_3}{}^{\beta_3} \\ & + \delta_{\alpha_1}{}^{\beta_2} \delta_{\alpha_2}{}^{\beta_3} \delta_{\alpha_3}{}^{\beta_1} - \delta_{\alpha_1}{}^{\beta_3} \delta_{\alpha_2}{}^{\beta_2} \delta_{\alpha_3}{}^{\beta_1} \\ & + \delta_{\alpha_1}{}^{\beta_3} \delta_{\alpha_2}{}^{\beta_1} \delta_{\alpha_3}{}^{\beta_2} - \delta_{\alpha_1}{}^{\beta_1} \delta_{\alpha_2}{}^{\beta_3} \delta_{\alpha_3}{}^{\beta_2} \Big] W_1^{\alpha_1}{}_{\beta_1} W_2^{\alpha_2}{}_{\beta_2} W_3^{\alpha_3}{}_{\beta_3} \\ = \; & \mathrm{Tr}\left(W_1\right) \mathrm{Tr}\left(W_2\right) \mathrm{Tr}\left(W_3\right) - \mathrm{Tr}\left(W_1 W_2\right) \mathrm{Tr}\left(W_3\right) \\ & + \mathrm{Tr}\left(W_1 W_2 W_3\right) - \mathrm{Tr}\left(W_1 W_3\right) \mathrm{Tr}\left(W_2\right) \\ & + \mathrm{Tr}\left(W_1 W_3 W_2\right) - \mathrm{Tr}\left(W_2 W_3\right) \mathrm{Tr}\left(W_1\right) \\ \equiv \; & 0 \end{aligned} \quad (3.10)$$

It is clear that there will be more and more constraint relations as the dimension of the lattice will increase. With the realization that there can be only one Mandelstam constraint in 2+1 dimension given in (3.3), the identity (3.10) must not be an independent

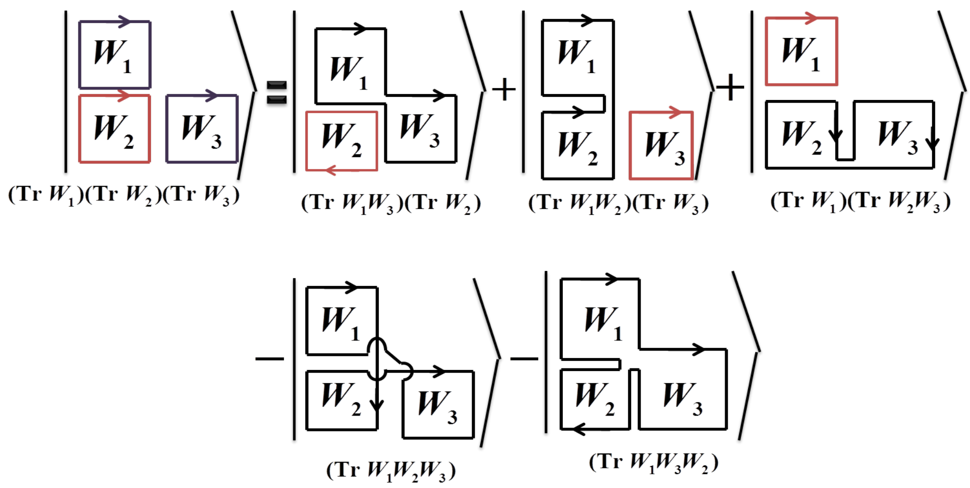

Figure 3.3: Mandelstam identities involving three loops originating and ending at one lattice site for SU(2) gauge theory.

one, but should just be a consequence of (3.3) and is obtained as below:

$$
\begin{aligned}
\text{R.H.S of (3.10)} \;=\; & \text{Tr}(W_1)\,\text{Tr}(W_2)\,\text{Tr}(W_3) - \text{Tr}(W_1W_2)\,\text{Tr}(W_3) \\
& + \underbrace{\text{Tr}(W_1W_2W_3)}_{\text{Tr}(W_1W_2)\,\text{Tr}(W_3) - \text{Tr}\left(W_1W_2W_3^{-1}\right)} - \text{Tr}(W_1W_3)\,\text{Tr}(W_2) \\
& + \underbrace{\text{Tr}(W_1W_3W_2)}_{\text{Tr}(W_1W_3)\,\text{Tr}(W_2) - \text{Tr}\left(W_1W_3W_2^{-1}\right)} - \text{Tr}(W_2W_3)\,\text{Tr}(W_1) \\
= \; & \text{Tr}(W_1)\,\text{Tr}(W_2)\,\text{Tr}(W_3) - \text{Tr}\left(W_1W_2W_3^{-1}\right) \\
& - \underbrace{\text{Tr}\left(W_1W_3W_2^{-1}\right)}_{\text{Tr}(W_1)\,\text{Tr}\left(W_3W_2^{-1}\right) - \text{Tr}(W_1)\,\text{Tr}\left(W_2W_3^{-1}\right)} - \text{Tr}(W_2W_3)\,\text{Tr}(W_1) \\
= \; & \text{Tr}(W_1)\,\text{Tr}(W_2)\,\text{Tr}(W_3) - \text{Tr}(W_1)\left(\text{Tr}(W_3W_2) + \text{Tr}\left(W_3W_2^{-1}\right)\right) \\
= \; & \text{Tr}(W_1)\,\text{Tr}(W_2)\,\text{Tr}(W_3) - \text{Tr}(W_1)\,\text{Tr}(W_2)\,\text{Tr}(W_3) = 0 \qquad (3.11)
\end{aligned}
$$

This general Mandelstam identity for $r = 3$ has also been illustrated in figure 3.3. It is practically impossible to solve all these constraints to find out the complete loop basis of the theory. That is why Mandelstam constraints have not been solved even for SU(2) case in arbitrary dimension and arbitrary lattice size and is still the main obstacle that one faces in the loop formulation of any gauge theory [21]. In [32] it is shown that the Mandelstam constraints constitute sufficient algebraic conditions on Wilson loop variables to allow reconstruction of the corresponding gauge potentials. In the past, the issues related to Mandelstam constraints have been mostly analyzed in the context of SU(2) gauge group, and that too within ad-hoc approximation schemes like small lattice/cluster size. For example, in [20] the Mandelstam constraints are solved and eigenvalues equations are analyzed on computer using small lattices and small loops. In [21, 22] an approximate loop cluster method in 2+1 dimensions is developed and the Schrödinger equation is expressed as difference equations in these cluster coordinates.

But from the SU(2) example we could see that the identities (3.8) corresponding to Migdal's Mandelstam constraints for $r = 3$ is not an independent one as it can be derived from another identity (3.3) involving only two Wilson loops. It also follows from similar calculation, that the identities involving any arbitrary number of loops $r = 4, 5, ...$ can also be derived using the single fundamental identity (3.3) for d=2.

In the next section we define loop states for SU(2) lattice gauge theory in terms of prepotentials and re-examine the above issues involving Mandelstam identities within the prepotential formulation.

3.2 Loop states in terms of Prepotentials: the non-abelian intertwining and abelian weaving

The advantage of the SU(2) prepotential operators is that under SU(2) gauge transformations they transform locally as SU(2) fundamental matter fields (2.13). Therefore, the SU(2) invariant loop Hilbert space $\mathcal{H}^L$ can be constructed and analyzed *locally* in terms of $\mathcal{H}^{SU(2)}(n)$ at each and every lattice sites n. Let us consider all the prepotentials around a lattice site n in a d dimensional lattice defined as : $a^{\dagger}_{\alpha}[n,i]$, $i = 1, 2, ..., 2d$, where we have omitted the left or right index as shown in Figure (3.4) for $d = 2$. Under the SU(2) gauge transformation all of these prepotentials transform as:

$$a^{\dagger}_{\alpha}[n,i] \rightarrow a^{\dagger}_{\beta}[n,i]\Lambda^{\dagger}_{\beta\alpha}(n), \quad i = 1, 2, .., 2d.$$

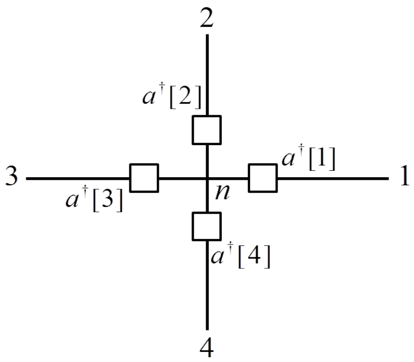

Figure 3.4: The 2d prepotential SU(2) doublets $a^\dagger[n,i]$, i=1,2,...,2d around every lattice site n shown in d = 2 by their Young tableau boxes □. They all transform as doublets under SU(2) gauge transformation at site n.

Hence, all possible SU(2) invariant operators at site n are the "intertwine" of any two different prepotentials belonging to two different links. i.e,

$$L_{ij}(n) = \epsilon_{\alpha\beta}\; a^\dagger_\alpha[n,i]\; a^\dagger_\beta[n,j] \equiv a^\dagger[n,i]\cdot\tilde{a}^\dagger[n,j], \qquad i,j=1,2,...,2d;\;\; i<j. \tag{3.12}$$

The ${}^{2d}C_2 = d(2d-1)$ intertwining operators $L_{ij}(n)$ in (3.12) correspond to putting two Young boxes corresponding to $[i]$ and $[j]$ directions into a single column by anti-symmetrizing them to construct SU(2) singlets. Intertwining between different prepotential operators has been shown in figure (3.5). Hence, the basic SU(2) invariant operators in our theory are the SU(2) Casimirs along with these intertwining operators which serves as the basic building blocks of any gauge invariant operator defined *locally* at each site of the lattice. These set of operators acting on the strong coupling vacuum create the complete SU(2) gauge invariant Hilbert space $\mathcal{H}^{SU(2)}(n)$ at site n. Note that $L_{ij}(n) = -L_{ji}(n)$, $L_{ii}=0$ implyies the fact that self intertwining is not allowed.

Thus a most general state in $\mathcal{H}^{\mathrm{SU}(2)}(n)$ is given by:

$$|\vec{l}(n)\,\rangle \equiv \left|\begin{array}{ccccc} l_{12} & l_{13} & l_{14} & .... & l_{1(2d)} \\ & l_{23} & l_{24} & .... & l_{2(2d)} \\ & & .... & .... & .... \\ & & & l_{2d-2(2d-1)} & l_{2d-2(2d)} \\ & & & & l_{2d-1(2d)} \end{array}\right\rangle = \prod_{\substack{i,j=1\\ j\rangle i}}^{2d} \left(L_{ij}(n)\right)^{l_{ij}}|0\rangle, \quad l_{ij}\in\mathcal{Z}_+. \tag{3.13}$$

In (3.13), $\mathcal{Z}_+$ denotes the set of all positive integers and $l_{ij}(n)(\equiv -l_{ji}(n), l_{ii}=0)$ are ${}^{2d}C_2$ SU(2) gauge invariant intertwining integer quantum numbers characterizing the SU(2)

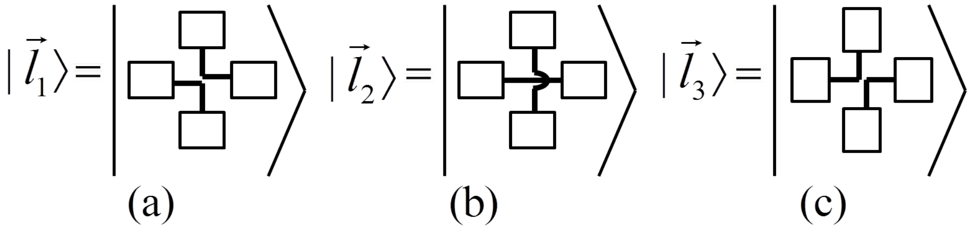

Figure 3.5: Graphical representation of SU(2) invariant intertwining illustrated for the states $|\vec{l}_1\rangle, |\vec{l}_2\rangle$ and $|\vec{l}_3\rangle$. The thick lines should be compared with the corresponding thick lines in Figure (3.1).

gauge invariant Hilbert space at the site n. The physical Hilbert space of gauge theory $\mathcal{H}^{SU(2)}_{phys}$ is obtained by taking direct product of $\mathcal{H}^{\mathrm{SU(2)}}(n)$ for all lattice sites and weaved by the U(1) Gauss law along each and every link. In Figure (3.5) we show the following three basic loop states constructed in terms of prepotentials in $d = 2$:

$$|\vec{l}_1\rangle = \left| \begin{array}{ccc} 1 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{array} \right\rangle, \quad |\vec{l}_2\rangle = \left| \begin{array}{ccc} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{array} \right\rangle, \quad |\vec{l}_3\rangle = \left| \begin{array}{ccc} 0 & 0 & 1 \\ & 1 & 0 \\ & & 0 \end{array} \right\rangle \tag{3.14}$$

The above three states $|\vec{l}_1\rangle = L_{12}L_{34}|0\rangle$, $|\vec{l}_2\rangle = L_{13}L_{24}|0\rangle$, $|\vec{l}_3\rangle = L_{14}L_{23}|0\rangle$ are manifestly local SU(2) gauge invariant as they commute with the Gauss law constraints. The states $|\vec{l}(n)\rangle$ are also eigenstates of individual $2d$ Casimirs along 2d links:

$$J[n,i].J[n,i]|\vec{l}(n)\rangle = j[n,i](j[n,i]+1)|\vec{l}(n)\rangle, \quad i = 1,2,3,4 \tag{3.15}$$

where,

$$2j[n,i] = \sum_{k\neq i=1}^{2d} l_{ik}(n), \quad l_{ik}(n) = l_{ki}(n), \quad l_{ik}(n) \in \mathcal{Z}_+ \tag{3.16}$$

We note that (3.16) is both necessary and sufficient condition on $j[n,i], i = 1,2,3,..,2d$ to get SU(2) singlets.

Note that, in terms of prepotentials, besides the SU(2) Gauss law, there also exists an additional U(1) Gauss law. The invariance under the abelian gauge transformation on links imply that on any link the number of left oscillators is equal to the number of right oscillators. Every link (n,i) satisfies this abelian Gauss law by carrying $N(n,i)$ units of abelian flux lines. Hence from the $SU(2) \otimes U(1)$ Gauss law the flux lines must be continuous throughout the lattice through the SU(2) intertwining at each of the sites. Thus, all possible SU(2) invariant intertwining within $\mathcal{H}_p^{\mathrm{SU(2)}}(n)$ and U(1) weaving of the neighboring $\mathcal{H}_p^{\mathrm{SU(2)}}(n')$ is geometrically equivalent to considering all possible loops on the lattice leading to the loop Hilbert space $\mathcal{H}_{phys}^{SU(2)}$, where n and n' denote any two neighbouring sites. that is:

$$\mathcal{H}_{phys}^{SU(2)} = \prod_n{}' \otimes \mathcal{H}_p^{\mathrm{SU(2)}}(n) \tag{3.17}$$

where, the prime over the product denote it to be consistent with the U(1) Gauss law along any link. Hence it turns out that these two picture of loops are mutually equivallent as from any given configuration of closed loops on a lattice one can find out all the intertwining quantum numbers $l_{ij}(n)$ at each site n within the loops by simply counting the number of loop lines going from $[i]^{th}$ to $[j]^{th}$ direction and vice versa. In the next section we discuss the issue of Mandelstam identities in detail and solve them explicitly in terms of the prepotential operators.

3.3 Mandelstam Constraints

In this section we discuss the issue of constraints, in context of the local loop basis $|\vec{l}(n)\rangle$ in (3.13) at lattice site n. The Wilson loop basis in terms of prepotentials in d spatial dimensions are locally characterized by (Number of intertwining quantum numbers per site $-$ Number of U(1) constraints per site) $= {}^{2d}C_2 - d = 2d(d-1)$ integers per lattice site. It is evident that this basis gives a overcomplete description of the physical Hilbert space of SU(2) gauge theory as there should be only $3(d-1)$ physical degrees of freedom per lattice site. It implies that there should be $2d(d-1) - 3(d-1)$ constraints present at each site. In $d = 2$, the number of such constraint is only one. We have already seen in terms of non-local Wilson loops, that there exists only one fundamental Mandelstam constraint for SU(2) as given in (3.3). This fact can be illustrated in a much better way using prepotentials. As we have seen in the last section that the loop operators existing at each lattice site are the $L_{ij}(n)$ operators defined in (3.12). The basic loop states satisfying the U(1) constraint at a site of a 2 dimensional lattice are $|\vec{l}_1\rangle$, $|\vec{l}_2\rangle$, $|\vec{l}_3\rangle$ given in (3.14).

Now, the six loop operators in $d = 2$ satisfies the identity:

$$(a^\dagger[1]\cdot\tilde{a}^\dagger[2])(a^\dagger[3]\cdot\tilde{a}^\dagger[4]) \equiv (a^\dagger[1]\cdot\tilde{a}^\dagger[3])(a^\dagger[2]\cdot\tilde{a}^\dagger[4]) - (a^\dagger[1]\cdot\tilde{a}^\dagger[4])(a^\dagger[2]\cdot\tilde{a}^\dagger[3]), \quad (3.18)$$

This identity is a consequence of the identity $\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$. The identity (3.18) implies that the states in (3.14) are linearly dependent:

$$|\vec{l}_1\ \rangle = |\vec{l}_2\ \rangle\ -\ |\vec{l}_3\ \rangle \quad (3.19)$$

Note that, *the SU(2) identity (3.3) involving link operators corresponds to the identity (3.18) and the Mandelstam constraint (3.6) is the constraint (3.19) written in terms of the relevant prepotential operators at site n.* Note that, in terms of the prepotentials, there exists only one identity in $d = 2$ in terms of prepotentials and that particular identity (3.18) is the only Mandelstam constraint present in the theory given in (3.3).

3.3.1 The solutions

After recasting Mandelstam identities in its local form using the prepotential operators, we will now solve them locally to find the orthonormal loop basis at each site n. Concentrating at a particular site n we have $2d$ angular momentum generators and hence $2d$ Casimirs, $J_i^2 \equiv \sum_{a=1}^{3} J^a[n,i]J^a[n,i], i = 1,2...,2d$, with eigenvalues $j_i(j_i+1)$. These set of operators are contained in the complete set of commuting observables (CSCO) at n, which are given by $4d$ angular momentum operators: $J_i^2, J_i^z, i = 1,2,..,2d$. Or equivalently we can also choose the complete set of mutually commuting operators [44, 57] as:

$$\begin{aligned} CSCO \ \equiv\ & \Big[J_1^2, ..J_{2d}^2; (J_1+J_2)^2, (J_1+J_2+J_3)^2 ..., \\ & \big(J_1+J_2+J_3...J_{(2d-1)}\big)^2, J_{total}^2, J_{total}^z\Big] \end{aligned} \quad (3.20)$$

with $J_{total}^2 = (J_1 + J_2^2 + J_3...... + J_{2d})^2$ and $J_{total}^z = (J_1^z + J_2^z + J_3^z...... + J_{2d}^z)$. Now, at each site, SU(2) Gauss law must hold implying $J_{total}^2 = J_{total}^z = 0$. Hence, the CSCO (3.20) is sufficient without last two operators in the list. Now, let us break the CSCO (3.20) into two parts:

$$\begin{aligned} CSCO(I) &= \left[J_1^2, J_2^2, .., J_{2d}^2\right], \\ CSCO(II) &= \Big[(J_1+J_2)^2, ..., \big(J_1+J_2+J_3......J_{(2d-1)}\big)^2 = 2d^2\Big] \end{aligned} \quad (3.21)$$

The corresponding SU(2) gauge invariant orthonormal eigenvectors are now characterized by [44, 57]

$$|j_1, j_2, ..j_{2d}; j_{12}, j_{123}, .., j_{12..(2d-1)} = j_{2d}\rangle \equiv |j_1, j_2, j_{12}, j_3, j_{123}, .., j_{2d-1}, j_{12..(2d-1)} = j_{2d}\rangle. \quad (3.22)$$

The states in (3.22) are characterized by the maximum possible $(4d-3)$ "good quantum numbers" which can be simultaneously measured at every lattice site.

The most general SU(2) loop states $|\vec{l}\,\rangle$ in (3.13) are eigenstates of of CSCO(I) with eigenvalues $j_i(j_i+1)$, where,

$$2j_i = \sum_{k=1}^{2d} l_{ik}.$$

Hence, these loop states can also be characterized by their angular momenta as:

$$|\,\vec{l}\,\rangle \equiv |j_1, j_2, ..., j_{2d}, j_{total} = m_{total} = 0\rangle \quad (3.23)$$

However, this characterization is not unique, as the j_i's do not fix l_{ij}'s uniquely. If we consider the loop state $|l\rangle$ on a 2 dimensional lattice, we find that under the following transformations of the loop quantum number, the angular momentum labeling remains same by (3.16):

$$\begin{aligned} &l_{12} \to l_{12} + r + t, \quad l_{13} \to l_{13} - r + s, \quad l_{14} \to l_{14} - s - t, \\ &l_{23} \to l_{23} - s - t, \quad l_{24} \to l_{24} - r + s, \quad l_{34} \to l_{34} + r + t. \end{aligned} \quad (3.24)$$

where, r, s, t can take all possible $\pm$ integer values such that $l_{i,j} \geq 0$. To be precise, these symmetries are the root cause of the Mandelstam constraints between the loop states. The loop states which are mutually dependent are those which are degenerate with respect to the CSCO(I). Or one can also say that the degenerate states with respect to CSCO(I) are all related by the Mandelstam identities (3.18). Therefore, we can lift the degeneracy and solve the Mandelstam constraints by demanding that the CSCO(I) degenerate eigenbasis (3.13) to be the eigenstates of CSCO(II) as well. A complete orthonormal loop basis in d dimension is locally characterized by $(4d-3)$ angular momentum quantum numbers and are given in (3.22). Among these, $2d$ are eigenvalues correponding to CSCO (I). The remaining $(2d-3)$ eigenvalues of CSCO(II) are not free and have to satisfy the triangular constraints:

$$|j_{12..(k-1)} - j_k| \leq j_{12..k} \leq j_{12..(k-1)} + j_k, \quad k = 2, 3, ..., (2d-1) \quad (3.25)$$

along with $j_{12..(2d-1)} = j_{2d}$.

3.3.2 The triangular constraints

The angular momentum labelling of the physical Hilbert space has been explored in literature in the context of duality transformation in lattice gauge theories [44] in terms of triangulated surfaces [57]. In this section, we discuss the solutions of the triangular constraints, which represents triangulated two dimensional surfaces together with intertwining or loop quantum numbers $(l_{12}, l_{13}, \ldots, l_{(2d-1)(2d)})$.

The SU(2) loop states characterized by $|j_1, j_2, ..., j_{2d}\rangle$ in the mapping (3.23) is consistent with the fact that, $\sum_k l_{ik} (= 2j_i)$, which gives the number of Young tableau boxes on the $[i]^{th}$ link. It intertwines with some other boxes in some k^{th} link, and the remaining flux is again available to intertwine with some other angular momentum. To illustrate this in our present scheme of angular momentum addition, Let us first consider $(1, 2)$ plane. To get the state $|j_1, j_2, ..., j_{2d}; j_{12}\rangle$ from the degenerate state $|j_1, j_2, ..., j_{2d}\rangle$, we need to intertwine (antisymmetrize) l_{12} boxes from $2j_1$ boxes with l_{12} boxes from $2j_2$ boxes so that we are left with $2j_{12}$ boxes in the (12) plane. Therefore, $2j_{12} = (2j_1 - l_{12}) + (2j_2 - l_{12})$. This process is sequential and can be repeated to get the eigenvalues of the CSCO(II) also in terms of the linking numbers:

$$\begin{aligned}
l_{12} &= j_1 + j_2 - j_{12} \\
l_{13} + l_{23} &= j_{12} + j_3 - j_{123} \\
l_{14} + l_{24} + l_{34} &= j_{123} + j_4 - j_{1234} \\
\vdots \quad &\quad \vdots \\
l_{1(2d)} + l_{2(2d)} + ... + l_{1(2d)} &= \underbrace{j_{12...(2d-1)}}_{=j_{2d}} + j_{2d} - \underbrace{j_{12...(2d)}}_{=0} = 2j_{2d}
\end{aligned} \tag{3.26}$$

Given j_{12} at a lattice site, the top equation fixes l_{12}, the next line fixes $l_{13} + l_{23}$ in terms of j_{12} and j_{123}, and so on. Together with (3.27), the following relations also hold for

d-dimensional lattice:

$$\begin{aligned}
2j_1 &= l_{12} + l_{13} + \ldots + l_{1(2d)} \\
2j_2 &= l_{23} + l_{24} + \ldots + l_{2(2d)} + l_{12} \\
2j_3 &= l_{34} + l_{35} + \ldots + l_{3(2d)} + l_{13} + l_{23} \\
&\vdots \\
2j_i &= \sum_{k=i+1}^{2d} l_{ik} + \sum_{k=1}^{i-1} l_{ki} \\
&\vdots \\
2j_{2d} &= l_{1(2d)} + l_{1(2d)} + \ldots + l_{(2d-1)(2d)}
\end{aligned} \tag{3.27}$$

Note that, the left hand side of each of the above set of equations gives the total number of Young tableaux boxes present on each of the $2d$ links meeting at the site. The last equations in (3.27) and (3.27) are exactly same, which is an identity as is already contained in (3.16).

In [29] all orthonormal loop states has been constructed in terms of prepotentials intertwining operators. SU(2) coherent states were used to construct the states together with the appropriate interpretation, modification and generalization of the techniques developed in [33]. We just quote the result below.

The orthonormal loop states for a d dimensional lattice are derived [29] as:

$$|LS\rangle_n \equiv |j_1, j_2, ..j_{2d}; j_{12}, j_{123}, ...j_{12..(2d-1)} = j_{2d}\rangle = N(j) \sum_{\{l\}}{}' \prod_{\substack{i,j \\ i<j}} \frac{1}{l_{ij}!} \big(L_{ij}(n)\big)^{l_{ij}(n)} |0\rangle \tag{3.28}$$

The prime over the summation means that, the linking numbers l_{ij} are are summed over all possible values which are consistent with (3.27) and (3.27). This summations is taken to construct orthonormal and complete basis out of the degenerate eigen states of CSCO(I). The normalization constant in (3.28) can be calculated exploiting the SU(2) coherent states and is obtained as [29]:

$$N(j) = N(j_1, j_2, j_{12}) N(j_{12}, j_3, j_{123}) N(j_{123}, j_4, j_{1234}) ......N(j_{2d}, j_{2d}, 0) \tag{3.29}$$

where $N(a, b, c) = \left[\frac{(2c+1)}{(a+b+c+1)!}\right]^{\frac{1}{2}} \left[(-a + b + c)!(a - b + c)!(a + b - c)!\right]^{\frac{1}{2}}$.

Considering the equivalence between the angular momentum quantum numbers and the linking quantum numbers in (3.28), we can find the origin of over-completeness of the loop basis as well as the Mandelstam identities explicitly. Let us first consider the $d = 2$ case. We see that the linking quantum numbers are related to the CSCO in the following way (3.27,3.27):

$$\begin{aligned}
2j_1 &= l_{12} + l_{13} + l_{14} \quad , \quad l_{12} = j_1 + j_2 - j_{12} \\
2j_2 &= l_{23} + l_{24} + l_{12} \quad , \quad l_{13} + l_{23} = j_{12} + j_3 - j_{123} \\
2j_3 &= l_{34} + l_{13} + l_{23} \quad , \quad l_{14} + l_{24} + l_{34} = \underbrace{j_{123}}_{=j_4} + j_4 - \underbrace{j_{1234}}_{=0}
\end{aligned} \tag{3.30}$$

This above set of relations imply that the linking quantum numbers l_{ij} and angular momentum labeling of the loop states are invariant under,

$$\begin{aligned}
&l_{12} \rightarrow l_{12} \ , \ l_{13} \rightarrow l_{13} + p \ , \ l_{14} \rightarrow l_{14} - p \\
&l_{23} \rightarrow l_{23} - p \ , \ l_{24} \rightarrow l_{24} + p \ , \ l_{34} \rightarrow l_{34}.
\end{aligned} \tag{3.31}$$

Note that these relations can as well be obtained from (3.24) only imposing the constraint that l_{12} is fixed. That implies $r = -t$ in (3.24), and $p = t + s$. Further note that, the loop states characterized by the linking quantum numbers, which are 6 in number for $d = 2$, are the overcomplete loop basis defined locally at each site of the lattice. The orthonormal loop hilbert space is spanned by the angular momentum basis $|j_1, j_2, j_3, j_4, j_{12}\rangle$. The mapping between these two basis is arbitrary upto a single parameter which denotes that there exist only one Mandelstam constraint between the overcomplete local loop states $|l\rangle$.

This analysis remains exactly same in any arbitrary dimension d. For example in $d = 3$, the orthonormal loop basis is characterized by ${}^6C_2 = 15$ quantum numbers. The corresponding angular momentum basis can also be obtained in the same way. Similar study also shows that the orthonormal basis is fixed with a set of symmetries in the liking quantum number. That symmetry is by 6 arbitrary parameters which denotes that there exist six fundamental Mandelstam constraints for $d = 3$. Let us also illustrate the example for $d = 3$ explicitly. The loop state characterized by linking quantum numbers is as follows:

$$|\vec{l}(n)\ \rangle \equiv \left| \begin{array}{ccccc} l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ & l_{23} & l_{24} & l_{25} & l_{26} \\ & & l_{34} & l_{35} & l_{36} \\ & & & l_{45} & l_{46} \\ & & & & l_{56} \end{array} \right\rangle = \prod_{\substack{i,j=1 \\ j\rangle i}}^{6} (L_{ij}(n))^{l_{ij}(n)} |0\rangle, \quad l_{ij}(n) \in \mathcal{Z}_{+}. \tag{3.32}$$

whereas, orthonormal loop Hilbert space is spanned by states:

$$|LS\rangle \equiv |j_1, j_2, j_{12}, j_3, j_{123}, j_4, j_{1234}, j_5, j_{12345} = j_6\rangle \tag{3.33}$$

The linking quantum numbers (3.32) represent the same loop state but are arbitrary upto the following transformations:

$$\begin{aligned} l_{12} &\rightarrow l_{12} \\ l_{13} &\rightarrow l_{13} + p_1 \\ l_{23} &\rightarrow l_{23} - p_1 \\ l_{14} &\rightarrow l_{14} + p_2 - p_3 \\ l_{24} &\rightarrow l_{24} + p_3 - p_4 \\ l_{34} &\rightarrow l_{34} + p_4 - p_2 \\ l_{15} &\rightarrow l_{15} + p_5 - p_6 \end{aligned}$$

$$\begin{aligned} l_{25} &\rightarrow l_{25} + p_6 - p_5 \\ l_{35} &\rightarrow l_{35} + 2p_5 - p_3 + p_2 \\ l_{45} &\rightarrow l_{45} - 2p_5 + p_3 - p_2 \\ l_{16} &\rightarrow l_{16} + p_6 - p_5 + p_3 - p_2 - p_1 \\ l_{26} &\rightarrow l_{26} + p_4 - p_6 + p_1 + p_5 - p_3 \\ l_{36} &\rightarrow l_{36} - p_5 - p_4 \\ l_{46} &\rightarrow l_{46} + 2p_5 - p_3 + p_2 \\ l_{56} &\rightarrow l_{56} \end{aligned} \tag{3.34}$$

compatible with the relations:

$$\begin{aligned}
2j_1 &= l_{12}+l_{13}+l_{14}+l_{15}+l_{16} \\
l_{12} &= j_1+j_2-j_{12} \\
2j_2 &= l_{23}+l_{24}+l_{25}+l_{26}+l_{12} \\
l_{13}+l_{23} &= j_{12}+j_3-j_{123} \\
2j_3 &= l_{34}+l_{35}+l_{36}+l_{13}+l_{23} \\
l_{14}+l_{24}+l_{34} &= j_{123}+j_4-j_{1234} \\
2j_4 &= l_{45}+l_{46}+l_{14}+l_{24}+l_{34} \\
l_{15}+l_{25}+l_{35}+l_{45} &= j_{1234}+j_5-j_6 \\
2j_5 &= l_{56}+l_{15}+l_{25}+l_{35}+l_{45} \\
l_{16}+l_{26}+l_{36}+l_{46}+l_{56} &= 2j_6 \qquad (3.35)
\end{aligned}$$

Note that, there is six arbitrary parameters $p_1, \ldots, p_6$ in the linking quantum number labeling which is equivalent to the six Mandelstam identities present for three dimensional SU(2) gauge theory.

The most important fact about the construction (3.28) is that the equivalent construction of orthonormal loop basis in terms of intertwining operators L_{ij} and intertwining linking numbers l_{ij} in arbitrary dimensions becomes extremely involved and complicated in terms of the link operators $U_{\alpha\beta}$ and the angular momentum quantum numbers. In that case the states

$$|j_1, j_2, j_{12}, j_3, j_{123}, ..., j_{2d-1}, j_{12..(2d-1)}, j_{total}, m_{total}\rangle$$

can be obtained by using Clebsch-Gordan coefficients:

$$\begin{aligned}
|j_1, j_2, j_{12}, j_3, j_{123}, ..., j_{12..(2d-1)} &= j_{2d}\rangle = \sum_{\vec{m}} C^{j_{12..2d}=0 m_{12..2d}=0}_{j_{12..(2d-1)}m_{12..(2d-1)}, j_{2d}m_{2d}} \\
&\quad ...C^{j_{123}m_{123}}_{j_{12}m_{12}, j_3m_3} C^{j_{12}m_{12}}_{j_1m_1, j_2m_2} \prod_{i=1}^{2d} \otimes |j_i m_i\rangle. \qquad (3.36)
\end{aligned}$$

However, in such approaches one must deal with gauge non-invariant Clebsch Gordan coefficients [58]. In this situation the only way out is found to be the use of graphical methods. In contrast, the construction (3.28) is in terms of gauge invariant intertwining numbers (not angular momentum) which makes it simple.

3.4 Summary and discussion

In this chapter we have discussed the loop formulation of lattice gauge theories emphasizing on the overcompleteness of the loop Hilbert space. We have illustrated in this chapter, how prepotential formulation enables us to overcome the problem of overcompleteness in loop Hilbert space and to construct complete and orthonormal basis for physical Hilbert space of Lattice gauge theory. For the purpose of illustration we took SU(2) group as an example. The full analysis consists of systematically solving Gauss law constraints and characterizing the states by linking quantum numbers, which solves the triangular constraints, then the solution of Mandelstam identities are obtained in terms of states characterized by angular momentum quantum numbers. Thus apparently highly non-local and formidable Mandelstam constraints in terms of the link operators were cast and then solved locally in terms of the prepotential intertwining operators. The manifestly SU(2) gauge invariant techniques involving gauge invariant local intertwining prepotential operators and intertwining/linking quantum numbers have direct geometrical interpretation in terms of loops. In the next chapter we consider the dynamical issues of the theory and explicitly calculate the dynamics of orthonormal loop states around a plaquette.

Chapter 4

Loops and Dynamics

As discussed in the previous chapter, the space of non-local Wilson loops is spanned by a overcomplete set of loop states satisfying the Mandelstam identities amongst themselves. The actual dynamics of the theory can only be obtained by studying the action of the Hamiltonian on all possible loop states. However, in the loop formulation of lattice gauge theories the dynamics of the Hamiltonian has been studied only within a truncated basis. The truncation of the loop basis is originaly motivated by the Strong coupling expansion, where only a few loops of smaller lengths and fluxes contribute to the spectrum. This understanding was applied to non-perturbative study of loops where the Hamiltonian is diagolnalized numerically within a truncated basis [18,20,22]. These small set of loops are considered to form a cluster, within which the dynamics is studied. For example, in [22], the cluster is considered to be consists of only three different loops constructed on the lattice. The orthogonal loops are a single plaquette, double winding of a plaquette and two adjacent plaquette with one link common. Clearly this approximation is quite crude and far from the real dynamics where all the loops interact among themselves via the action of the Hamiltonian. Another similar attempt [18] to understand the dynamics of the gauge theory involved the study of the lattice Schrodinger equation. In this work the eigenvalue equation is mapped to a discretized version of Mathieu equation, but again with an abrupt trucnation of the physical basis. In 3+1 dimension the orthogonal loop states created from vacuum by upto three actions of the Hamiltonian [18], i.e loops involved with three plaquettes which are 39 in number was considered to study the deformation of the loops by the action of Hamiltonian and finally to study the spectrum. The Hamiltonian was diagonalized numerically within this basis and mass gap was calculated. Another work to be mentioned is [20] where, all the orthogonal loops upto a maximum length of 12 lattice units existing on a 4×4 lattice were considered. The number of possible loops actually grows exponentially with increasing length. Hence it becomes computationally tough to solve the Schrodinger equation within that basis. In [20], the Hamiltonian is solved within the basis spanned by 8660 independent loop states. It is worth mentioning that the Hamiltonian matrix written in loop basis is a sparse one, i.e most of the matrix elements are zero as only a few loops interact among themselves through the single action of Hamiltonian.

However we have already seen how the prepotential formulation enables us to construct the exact and orthonormal loop basis present locally at each site of the lattice. Having constructed the orthonormal loop basis at each site, we are now in a platform to calculate the loop dynamics in terms of prepotentials.

4.1 Dynamics within local orthonormal loop states

To discuss dynamics of loops, we consider pure SU(2) lattice gauge theory Hamiltonian [2]:

$$H = \sum_{n,i}\sum_{\mathrm{a}=1}^{3} E^{\mathrm{a}}(n,i)E^{\mathrm{a}}(n,i) + K\sum_{\square} Tr\left(U_{\square} + U_{\square}^{\dagger}\right) \tag{4.1}$$

For calculational simplicity we consider the lattice to be in dimension two in the beginning. Let us consider a plaquette *abcd* as shown in the Figure (4.1). Using (2.24), we write the gauge invariant $\mathrm{Tr}U_{\mathrm{plaquette}}$ over *abcd* in terms of the prepotentials:

$$\begin{aligned}
&\mathrm{T}rU_{abcd} = F_{abcd}\Big[\left(a^{\dagger}[1]\cdot\tilde{a}^{\dagger}[2]\right)_a\left(a^{\dagger}[2]\cdot\tilde{a}^{\dagger}[3]\right)_b\left(a^{\dagger}[3]\cdot\tilde{a}^{\dagger}[4]\right)_c\left(a^{\dagger}[4]\cdot\tilde{a}^{\dagger}[1]\right)_d \\
&+\sum_{i=1}^{4}\pi(l_i) + \sum_{i,j>i=1}^{4}\pi(l_i)\pi(l_j) + \sum_{i,j>i,k>j=1}^{4}\pi(l_i)\pi(l_j)\pi(l_k) + \pi(l_1)\pi(l_2)\pi(l_3)\pi(l_4)\Big]F_{abcd} \\
&\equiv \sum_{\alpha\beta\gamma\delta=\pm} H_{\alpha\beta\gamma\delta}
\end{aligned} \tag{4.2}$$

where $F_{abcd} \equiv F(l_1)F(l_2)F(l_3)F(l_4)$. In (4.2), there are sixteen $SU(2)\otimes U(1)$ gauge invariant terms which are produced by substituting (2.24) in the $\mathrm{Tr}(U_{plaquette})$ term of the

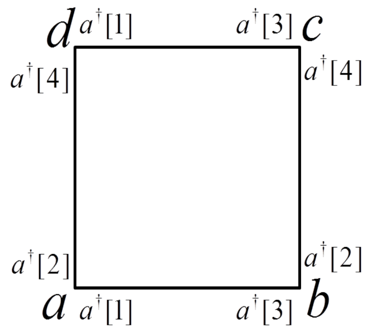

Figure 4.1: The plaquette **abcd** with the corresponding prepotential operators from Figure (3.4).

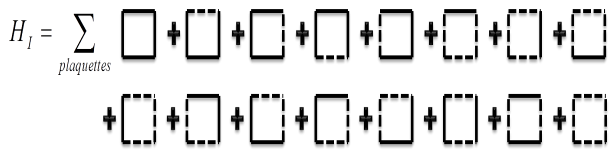

Figure 4.2: Pictorial representation of (4.2). The solid line along a link implies increase of flux along that link whereas dotted line means decrease of flux.

Hamiltonian. The first plaquette operator on the r.h.s. of the top equation in (4.2) is written explicitly in terms of the prepotential intertwining operators at lattice sites **a**, **b**, **c** and **d** in Figure (4.1). The action of the first term is to increase the angular momentum flux by $\frac{1}{2}$ units on all of the four links of *abcd* and therefore we represent it by H_{++++}. The other 15 terms changes the flux over the plaquette in all possible way keeping the U(1) Gauss law valid for each link. All those actions can be understood by the action of π. The single π operation at any of the 4 links (l_1, l_2, l_3, l_4) acting on H_{++++} produces 4 terms: $H_{-+++}, H_{+-++}, H_{++-+}, H_{+++-}$. Similarly, the double π operation produces 6 operators: $H_{--++}, H_{-+-+}, H_{-++-}, H_{+--+}, H_{+-+-}, H_{++--}$. There are four 3 π terms in (4.2): $H_{+---}, H_{-+--}, H_{--+-}, H_{---+}$. Finally, the 4 π operation on all the four links produces a single term: H_{----}. Note that $\left(H_{\alpha\beta\gamma\delta}\right)^\dagger = H_{-\alpha-\beta-\gamma-\delta}$. This decomposition is pictorially illustrated in figure 7.40 where, the dotted line denotes the π operation on that link. The advantage of the form (4.2) is that now this magnetic field term can be analyzed locally at four sites surrounding the plaquette **a**, **b**, **c** and **d** in the manifestly SU(2) gauge invariant way. The loop state around a plaquette $|j_{abcd}\rangle$ can be written as

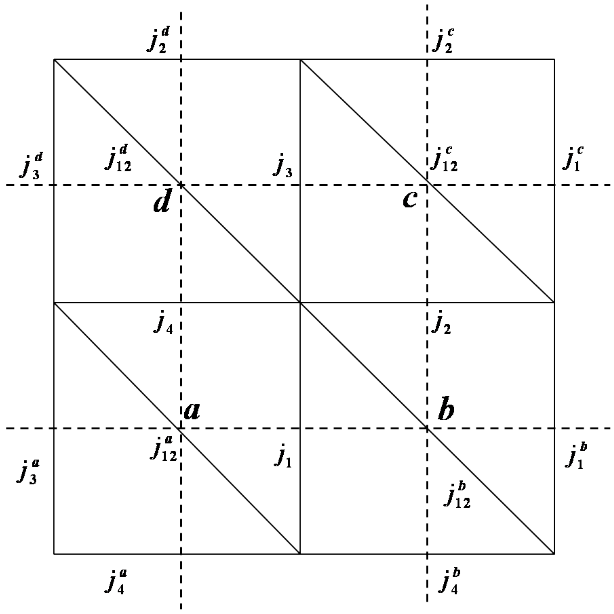

Figure 4.3: Angular momentum addition scheme around a plaquette

(see figure 4.3):

$$|j_{abcd}\rangle \equiv |j_1^a j_2^a j_3^a j_4^a j_{12}^a\rangle \otimes |j_1^b j_2^b j_3^b j_4^b j_{12}^b\rangle \otimes |j_1^c j_2^c j_3^c j_4^c j_{12}^c\rangle \otimes |j_1^d j_2^d j_3^d j_4^d j_{12}^d\rangle. \tag{4.3}$$

To satiafy the U(1) Gauss law (see Figure (4.3)):

$$j_1^a = j_3^b \equiv j_1,\ j_2^b = j_4^c \equiv j_2,\ j_3^c = j_1^d \equiv j_3,\ j_4^d = j_2^a \equiv j_4. \tag{4.4}$$

We now compute the matrix elements of $\mathrm{T}rU_{abcd}$ directly within the loop basis (4.3) in the next section using generalized Wigner-Eckart theorem and Biedenharn-Elliot identities.

4.2 Explicit Computation in 2 dimension

The electric part of the Hamiltonian (4.1) simply counts the abelian flux lines without changing the loop states. The action of the magnetic field term in (4.2) on the loop states is non-trivial. We now calculate the matrix elements of $\mathrm{Tr}U_{\Box}$ in d=2 below.

Let us define the square root of the multiplicity factors and phase factors:

$$\Pi(x, y, ...) \equiv \sqrt{(2x+1)(2y+1)....}, \quad (4.5)$$
$$\eta(x, y) \equiv (-1)^{x+y+\frac{1}{2}} \quad (4.6)$$

for later notational convenience.

The matrix elements of an intertwining operators $L_{12} = \left(a^\dagger[1]\cdot\tilde{a}^\dagger[2]\right)$ in the corresponding angular momentum basis $|j_1, j_2, j_{12}, m_{12}\rangle$ are given by the generalized Wigner Eckart theorem[1] [38, 39]:

$$\begin{aligned}
&\langle \bar{j}_1, \bar{j}_2, \bar{j}_{12}, \bar{m}_{12}| \left(a^\dagger[1]\cdot\tilde{a}^\dagger[2]\right) |j_1, j_2, j_{12}, m_{12}\rangle \\
&= \sqrt{2}\ \langle \bar{j}_1, \bar{j}_2, \bar{j}_{12}, \bar{m}_{12}| \left(a^\dagger[1]\otimes a^\dagger[2]\right)^0_0 |j_1, j_2, j_{12}, m_{12}\rangle \\
&= \sqrt{2}\ (-1)^{(\bar{j}_{12}-\bar{m}_{12})}\ \Pi(j_{12}, \bar{j}_{12}, 0) \begin{pmatrix} \bar{j}_{12} & 0 & j_{12} \\ -\bar{m}_{12} & 0 & m_{12} \end{pmatrix} \begin{Bmatrix} \bar{j}_1 & j_1 & \frac{1}{2} \\ \bar{j}_2 & j_2 & \frac{1}{2} \\ \bar{j}_{12} & j_{12} & 0 \end{Bmatrix} \\
&\times\ \langle \bar{j}_1||a^\dagger[1]||j_1\rangle\langle \bar{j}_2||a^\dagger[2]||j_2\rangle \qquad (4.7)
\end{aligned}$$

In (4.7), $\left(a^\dagger[1]\otimes a^\dagger[2]\right)^0_0 \equiv \sum_{m,\bar{m}=\pm\frac{1}{2}} C^{0,0}_{\frac{1}{2},m;\frac{1}{2},\bar{m}} a^\dagger_m b^\dagger_{\bar{m}}$ with $a_{+\frac{1}{2}} \equiv a_1$, $a_{-\frac{1}{2}} \equiv a_2$. The reduced matrix elements of the prepotential operators are given by[2]

$$\langle \bar{j}||a||j\rangle = \Pi(j, \bar{j})\ \delta_{\bar{j}, j-\frac{1}{2}}, \qquad \langle \bar{j}||a^\dagger||j\rangle = \Pi(j, \bar{j})\ \delta_{\bar{j}, j+\frac{1}{2}}. \quad (4.8)$$

---

[1] The WignerEckart theorem is a theorem of representation theory and quantum mechanics. It states that matrix elements of spherical tensor operators on the basis of angular momentum eigenstates can be expressed as the product of two factors, one of which is independent of angular momentum orientation (the reduced matrix element), and the other a Clebsch-Gordan coefficient. The theorem reads:

$$\langle j, m|T^k_q|j'm'\rangle = \langle j||T^k||j'\rangle C^{jm}_{kqj'm'}$$

where T^k_q is a rank k spherical tensor, $|jm\rangle$ and $|j'm'\rangle$ are eigenkets of total angular momentum J^2 and its z-component J_z, $\langle j||T^k||j'\rangle$ is the reduced matrix element having a value which is independent of m and q, and $C^{jm}_{kqj'm'} = \langle j'm'; kq|jm\rangle$ is the Clebsch-Gordan coefficient for adding j' and k to get j.

In effect, the WignerEckart theorem says that operating with a spherical tensor operator of rank k on an angular momentum eigenstate is like adding a state with angular momentum k to the state. The matrix element one finds for the spherical tensor operator is proportional to a Clebsch-Gordan coefficient, which arises when considering adding two angular momenta.

[2] Reduced matrix element of any operator O^J_M is calculated as:

$$\langle j'||O^J_M||j\rangle = \frac{2J+1}{2j'+1}\sum_{m',m} C^{j'm'}_{j,m;J,M}\langle j'm'|O^J_M|jm\rangle$$

Note, $J = 1/2, M = \pm 1/2$ for $a^\dagger$ and $J = -1/2, M = \mp 1/2$ for a.

The coefficients $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$, $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix}$ represent the $3j$ and $9j$ symbols respectively. Using the values [38]:

$$\begin{aligned} \begin{pmatrix} \bar{j}_{12} & 0 & j_{12} \\ -\bar{m}_{12} & 0 & m_{12} \end{pmatrix} &= \frac{(-1)^{-\bar{j}_{12}+\bar{m}_{12}}}{\Pi(j_{12})} \delta_{j_{12},\bar{j}_{12}}\delta_{m_{12},\bar{m}_{12}}, \\ \begin{Bmatrix} \bar{j}_1 & j_1 & \frac{1}{2} \\ \bar{j}_2 & j_2 & \frac{1}{2} \\ \bar{j}_{12} & j_{12} & 0 \end{Bmatrix} &= \frac{(-1)^{j_1+\frac{1}{2}+\bar{j}_2+\bar{j}_{12}}}{\Pi(\frac{1}{2}, j_{12})} \delta_{j_{12},\bar{j}_{12}} \begin{Bmatrix} \bar{j}_1 & j_1 & \frac{1}{2} \\ j_2 & \bar{j}_2 & j_{12} \end{Bmatrix}, \end{aligned}$$

we get:

$$\begin{aligned} \langle \bar{j}_1, \bar{j}_2, \bar{j}_{12}, \bar{m}_{12} | L_{12} | j_1, j_2, j_{12}, m_{12} \rangle &= \delta_{j_{12},\bar{j}_{12}}\delta_{m_{12},\bar{m}_{12}}(-1)^{j_{12}}\eta(j_1,\bar{j}_2) \begin{Bmatrix} \bar{j}_1 & j_1 & \frac{1}{2} \\ j_2 & \bar{j}_2 & j_{12} \end{Bmatrix} \\ & \quad \langle \bar{j}_1 || a^\dagger[1] || j_1 \rangle \langle \bar{j}_2 || a^\dagger[2] || j_2 \rangle. \end{aligned} \tag{4.9}$$

Note that (4.9) can also be checked by directly applying the above intertwining operator on the loop basis to get it's matrix elements (4.9) algebraically. It is clear that the intertwining operator $L_{12} = a^\dagger[1] \cdot \tilde{a}^\dagger[2]$ increases the fluxes j_1 and j_2 by $\frac{1}{2}$ units each. Further, as L_{12} commutes with the $(J[1] + J[2])^2$, the matrix elements are diagonal in j_{12} and m_{12}. The matrix elements of the intertwining operators in the geometrical form (4.9) also tell us that all the 16 terms in $\sum_{\alpha\beta\gamma\delta=\pm} H_{\alpha\beta\gamma\delta}$ in (4.2) differ only in their reduced matrix element structures. *Therefore, we need to compute the matrix elements of only a single term in (4.2), providing enormous simplification at the algebraic level.* Let us choose this term to be the first term in (4.2) associated with the plaquette *abcd* in Figure (4.1):

$$H_{++++} \equiv F_{abcd} \left(a^\dagger[1] \cdot \tilde{a}^\dagger[2]\right)_a \left(a^\dagger[2] \cdot \tilde{a}^\dagger[3]\right)_b \left(a^\dagger[3] \cdot \tilde{a}^\dagger[4]\right)_c \left(a^\dagger[4] \cdot \tilde{a}^\dagger[1]\right)_d F_{abcd} \tag{4.10}$$

In computing the loop dynamics below, it is required to change the angular momentum addition scheme suitably,

$$\begin{aligned} |j_1, j_2, j_3, j_4, j_{12}\rangle &\equiv |j_1, j_2, j_{12}, j_3, j_{123}(= j_4), j_4, j_{total} = j_{(123)(4)} = 0\rangle \\ = |(j_1, j_2)j_{12}, (j_3, j_4)j_{34}(= j_{12}), j_{total} &= j_{(12)(34)} = 0\rangle \equiv |(j_1, j_2)j_{12}, (j_3, j_4)j_{12}\rangle \end{aligned} \tag{4.11}$$

The equivalent scheme on the right of (4.11) simplifies the algebra. The relation (4.11) is obtained by writing:

$$|j_1,j_2,j_3,j_4,j_{12}\rangle \equiv \sum_{all\ m} C^{\ \ 0,0}_{j_{123}m_{123},j_4m_4} C^{j_{123},m_{123}}_{j_{12}m_{12},j_3m_3}|j_1,j_2,j_{12},m_{12}\rangle|j_3m_3\rangle|j_4m_4\rangle$$
$$= \sum_{all\ m}\sum_{j_{34}} C^{0,0}_{j_{12}m_{12},j_{34}m_{34}}|j_1,j_2,j_{12},m_{12}\rangle|j_3,j_4,j_{34},m_{34}\rangle = |(j_1,j_2)j_{12},(j_3,j_4)j_{12}\rangle.$$

We have used:

$$C^{\ \ 0,0}_{j^a_{123}m^a_{123},j^a_4m^a_4} = \frac{(-1)^{j^a_4+m^a_4}}{\Pi(j^a_4)}\delta_{j^a_{123},j^a_4}\delta_{m^a_{123},-m^a_4},$$
$$\frac{(-1)^{j^a_4+m^a_4}}{\Pi(j^a_4)}C^{j^a_4-m^a_4}_{j^a_{12}m^a_{12},j^a_3m^a_3} = C^{0,0}_{j^a_{12}m^a_{12},j^a_{12}-m^a_{12}}C^{j^a_{12}-m^a_{12}}_{j^a_3m^a_3,j^a_4m^a_4}.$$

Note that the normalization operator F_{abcd} in (4.2) acting on the loop states gives $|j_{abcd}\rangle$ defined in (4.3)and (4.4):

$$F_{abcd}|j_{abcd}\rangle = \frac{1}{\Pi(j_1,j_2,j_3,j_4)}|j_{abcd}\rangle \tag{4.12}$$

Therefore, we only need to compute the matrix elements of the intertwining operators in (4.10) in the orthonormal loop basis $|j_1,j_2,j_3,j_4,j_{12}$.

Loop dynamics at a:

In H_{++++} above, the intertwining operator at a is $\left(a^\dagger[1]\cdot\tilde{a}^\dagger[2]\right)_a$. Using (4.9), one directly gets:

$$\langle\bar{j}^a_1,\bar{j}^a_2,\bar{j}^a_3,\bar{j}^a_4,\bar{j}^a_{12}|(a^\dagger[1]\cdot\tilde{a}^\dagger[2])_a|j^a_1,j^a_2,j^a_3,j^a_4,j^a_{12}\rangle = (-1)^{j^a_{12}}\eta(j^a_1,\bar{j}^a_2)\delta_{j^a_3,\bar{j}^a_3}\delta_{j^a_4,\bar{j}^a_4}\delta_{j^a_{12},\bar{j}^a_{12}}$$
$$\times\begin{Bmatrix} j^a_1 & \bar{j}^a_1 & \frac{1}{2} \\ \bar{j}^a_2 & j^a_2 & j^a_{12}\end{Bmatrix}\langle\bar{j}^a_1||a^\dagger[1]||j^a_1\rangle\langle\bar{j}^a_2||a^\dagger[2]||j^a_2\rangle \tag{4.13}$$

The intertwining operator $\left(a^\dagger[1]\cdot\tilde{a}^\dagger[2]\right)_a$ increases the SU(2) flux on the links 1 and 4 of Figure (4.1). Note that this information is contained only in the last two reduced matrix element terms in (4.13).

Loop dynamics at b:

The intertwining operator at b in (4.10) is $\left(a^\dagger[2]\cdot\tilde{a}^\dagger[3]\right)_b$. To compute it's action at b, we write the loop states (3.28) in terms of the basis states which diagonalize $(J[2]+J[3])^2$.

This is done by changing the angular momentum coupling scheme at site 'b' by the following relation [38]:

$$
\begin{aligned}
|j_1^b j_2^b j_3^b j_4^b j_{12}^b\rangle &= (-1)^{(j_1^b+j_2^b+j_3^b+j_{123}^b)} \sum_{j_{23}^b} \Pi(j_{12}^b, j_{23}^b) \begin{Bmatrix} j_1^b & j_2^b & j_{12}^b \\ j_3^b & j_{123}^b & j_{23}^b \end{Bmatrix} |j_1^b j_2^b j_3^b j_4^b j_{23}^b\rangle \\
&= (-1)^{(j_1^b+j_2^b+j_3^b+j_4^b)} \sum_{j_{23}^b} \Pi(j_{12}^b, j_{23}^b) \begin{Bmatrix} j_1^b & j_2^b & j_{12}^b \\ j_3^b & j_4^b & j_{23}^b \end{Bmatrix} |j_1^b j_2^b j_3^b j_4^b j_{23}^b\rangle
\end{aligned}
\tag{4.14}
$$

In (4.14), $|j_1^b j_2^b j_3^b j_4^b j_{23}^b\rangle \equiv |j_1^b, (j_2^b j_3^b), j_{23}^b, j_{123}^b = j_4^b, j_{1234}^b = m_{1234}^b = 0\rangle$. Note that the phase factor $(-1)^{(j_1^b+j_2^b+j_3^b+j_4^b)}$ in (4.14) is real because of the triangular constraints on the angular momenta or equivalently (3.16). Now we use:

$$
|j_1^b j_2^b j_3^b j_4^b j_{23}^b\rangle = (-1)^{2j_1^b} |j_2^b j_3^b j_4^b j_1^b j_{23}^b = j_{41}^b\rangle \tag{4.15}
$$

and (4.9) to get:

$$
\langle \bar{j}_1^b, \bar{j}_2^b, \bar{j}_3^b, \bar{j}_4^b, \bar{j}_{12}^b | \left(a^\dagger[2] \cdot \tilde{a}^\dagger[3]\right)_b |j_1^b, j_2^b, j_3^b, j_4^b, j_{12}^b\rangle = (-1)^{j_2^b+j_3^b-\bar{j}_2^b-\bar{j}_3^b} \eta(j_2^b, \bar{j}_3^b) \delta_{j_1^b,\bar{j}_1^b} \delta_{j_4^b,\bar{j}_4^b} \Pi(j_{12}^b, \bar{j}_{12}^b)
$$

$$
\sum_{j_{23}^b} (-)^{j_{23}^b} \Pi^2(j_{23}^b) \begin{Bmatrix} j_3^b & j_2^b & j_{23}^b \\ j_1^b & j_4^b & j_{12}^b \end{Bmatrix} \begin{Bmatrix} j_1^b & \bar{j}_4^b & j_{23}^b \\ \bar{j}_3^b & j_2^b & \bar{j}_{12}^b \end{Bmatrix} \begin{Bmatrix} \bar{j}_3^b & \bar{j}_2^b & j_{23}^b \\ j_2^b & j_3^b & \frac{1}{2} \end{Bmatrix} \langle \bar{j}_2^b || a^\dagger[2] || j_2^b \rangle \langle \bar{j}_3^b || a^\dagger[3] || j_3^b \rangle
$$

The summation over j_{23}^b in the last line above can be performed using Biedenharn-Elliot identity [38]:

$$
\sum_x (-1)^x \Pi^2(x) \begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix} \begin{Bmatrix} c & d & x \\ e & f & q \end{Bmatrix} \begin{Bmatrix} e & f & x \\ b & a & s \end{Bmatrix} = (-1)^{-r} \begin{Bmatrix} p & q & s \\ e & a & d \end{Bmatrix} \begin{Bmatrix} p & q & s \\ f & b & c \end{Bmatrix}
$$
$$
r = (a+b+c+d+e+f+p+q+s).
$$

Finally, the loop dynamics at lattice site b is given by:

$$
\langle \bar{j}_1^b, \bar{j}_2^b, \bar{j}_3^b, \bar{j}_4^b, \bar{j}_{12}^b | \left(a^\dagger[2] \cdot \tilde{a}^\dagger[3]\right)_b |j_1^b, j_2^b, j_3^b, j_4^b, j_{12}^b\rangle = (-1)^{j_1^b+j_4^b} \eta(j_2^b, \bar{j}_3^b) \eta(j_{12}^b, \bar{j}_{12}^b) \delta_{j_1^b,\bar{j}_1^b} \delta_{j_4^b,\bar{j}_4^b}
$$

$$
\Pi(j_{12}^b, \bar{j}_{12}^b) \begin{Bmatrix} j_{12}^b & \bar{j}_{12}^b & \frac{1}{2} \\ \bar{j}_3^b & j_3^b & j_4^b \end{Bmatrix} \begin{Bmatrix} j_{12}^b & \bar{j}_{12}^b & \frac{1}{2} \\ \bar{j}_2^b & j_2^b & j_1^b \end{Bmatrix} \langle \bar{j}_2^b || a^\dagger[2] || j_2^b \rangle \langle \bar{j}_3^b || a^\dagger[3] || j_3^b \rangle \tag{4.16}
$$

Note that the intertwining operator $\left(a^\dagger[2] \cdot \tilde{a}^\dagger[3]\right)_b$ changes the flux in the links 1 and 2 as it is clear from the reduced matrix elements as (4.4).

Loop dynamics at c:

The suitable angular momentum coupling scheme to calculate the matrix element of the operator $\left(a^\dagger[3]\cdot\tilde{a}^\dagger[4]\right)_c$ is $|j_1, j_2, j_{12}, j_3, j_4, j_{34} = j_{12}, j_{1234} = 0, m_{1234} = 0\rangle$. Hence we use 4.11 to write At c, we use (4.11) to write

$$|j_1^c, j_2^c, j_3^c, j_4^c, j_{12}^c\rangle = |(j_1^c, j_2^c)j_{12}^c, (j_3^c j_4^c), j_{12}^c\rangle = (-1)^{2j_{12}^c}|(j_3^c, j_4^c)j_{12}^c, (j_1^c j_2^c)j_{12}^c\rangle$$

to get:

$$\langle \bar{j}_1^c, \bar{j}_2^c, \bar{j}_3^c, \bar{j}_4^c, \bar{j}_{12}^c|\left(a^\dagger[3]\cdot\tilde{a}^\dagger[4]\right)_c|j_1^c, j_2^c, j_3^c, j_4^c, j_{12}^c\rangle = (-1)^{j_{12}^c}\eta(j_3^c, \bar{j}_4^c)\delta_{j_1^c,\bar{j}_1^c}\delta_{j_2^c,\bar{j}_2^c}\delta_{j_{12}^c,\bar{j}_{12}^c}$$
$$\left\{\begin{array}{ccc} j_3^c & \bar{j}_3^c & \frac{1}{2} \\ \bar{j}_4^c & j_4^c & j_{12}^c \end{array}\right\}\langle \bar{j}_3^c||a^\dagger[3]||j_3^c\rangle\langle \bar{j}_4^c||a^\dagger[4]||j_4^c\rangle \quad (4.17)$$

Again note that the change in the flux is on links 2 and 3 from the reduced matrix elements and equation (4.4).

Loop dynamics at d:

In this case the $a^\dagger[4]\cdot\tilde{a}^\dagger[1]$ is easily computed within the basis $|j_1 j_2 j_3 j_{23} j_4 j_{41} = j_{23}\rangle$. Hence to compute the loop dynamics at d, we write:

$$|j_1^d, j_2^d, j_3^d, j_4^d, j_{12}^d\rangle = |j_1^d, j_2^d, j_{12}^d, j_3^d, j_4^d, j_{34}^d(= j_{12}^d)\rangle$$
$$= \sum_{j_{14}^d}(-1)^{j_3^d+j_4^d-j_{12}^d}\Pi(j_{12}^d, j_{34}^d, j_{14}^d, j_{23}^d)\left\{\begin{array}{ccc} j_1^d & j_2^d & j_{12}^d \\ j_4^d & j_3^d & j_{34}^d = j_{12}^d \\ j_{14}^d & j_{23}^d = j_{14}^d & 0 \end{array}\right\}|j_4^d, j_1^d, j_{14}^d, j_2^d, j_3^d, j_{23}^d\rangle$$
$$= (-1)^{2j_4^d}\sum_{j_{14}}\Pi(j_{12}^d, j_{14}^d)(-1)^{j_1^d+j_2^d+j_3^d+j_4^d}\left\{\begin{array}{ccc} j_1^d & j_2^d & j_{12}^d \\ j_3^d & j_3^d & j_{14}^d \end{array}\right\}|j_4^d, j_1^d, j_{14}^d, j_2^d, j_3^d, j_{23}^d\rangle \quad (4.18)$$

In (4.18), we have used

$$\left\{\begin{array}{ccc} j_1^d & j_2^d & j_{12}^d \\ j_4^d & j_3^d & j_{12}^d \\ j_{14}^d & j_{14}^d & 0 \end{array}\right\} = (-1)^{j_2^d+j_{12}^d+j_4^d+j_{14}^d}\left(\Pi(j_{12}^d, j_{14}^d)\right)^{-1}\left\{\begin{array}{ccc} j_1^d & j_2^d & j_{12}^d \\ j_3^d & j_3^d & j_{14}^d \end{array}\right\}.$$

Finally, using (4.18) and the Biedenharn-Elliot identity, the dynamics at d is given by:

$$\langle \bar{j}_1^d, \bar{j}_2^d, \bar{j}_3^d, \bar{j}_4^d, \bar{j}_{12}^d|\left(a^\dagger[4]\cdot\tilde{a}^\dagger[1]\right)_d|j_1^d, j_2^d, j_3^d, j_4^d, j_{12}^d\rangle = -(-1)^{j_2^d+j_3^d}\eta(j_4^d, \bar{j}_1^d)\eta(j_{12}^d, \bar{j}_{12}^d)$$

$$\delta_{j_2^d,\bar{j}_2^d}\delta_{j_3^d,\bar{j}_3^d}\Pi(j_{12}^d, \bar{j}_{12}^d)\left\{\begin{array}{ccc} j_{12}^d & \bar{j}_{12}^d & \frac{1}{2} \\ \bar{j}_1^d & j_1^d & j_2^d \end{array}\right\}\left\{\begin{array}{ccc} j_{12}^d & \bar{j}_{12}^d & \frac{1}{2} \\ \bar{j}_4^d & j_4^d & j_3^d \end{array}\right\}\langle \bar{j}_4^d||a^\dagger[4]||j_4^d\rangle\langle \bar{j}_1^d||a^\dagger[1]||j_1^d\rangle \quad (4.19)$$

Again note that the change in the flux is on links 3 and 4 from the reduced matrix elements and relation (4.4).

Loop dynamics at abcd:

As stated while computing dynamics at individual site,

$$\begin{aligned}
\langle \bar{j}^a_1 || a^\dagger[1] || j^a_1 \rangle = \langle \bar{j}^b_3 || a^\dagger[3] || j^b_3 \rangle = \Pi(j_1, \bar{j}_1)\delta_{\bar{j}_1, j_1+\frac{1}{2}}, \\
\langle \bar{j}^b_2 || a^\dagger[2] || j^b_2 \rangle = \langle \bar{j}^c_4 || a^\dagger[4] || j^c_4 \rangle = \Pi(j_2, \bar{j}_2)\delta_{\bar{j}_2, j_2+\frac{1}{2}}, \\
\langle \bar{j}^c_3 || a^\dagger[3] || j^c_3 \rangle = \langle \bar{j}^d_1 || a^\dagger[1] || j^d_1 \rangle = \Pi(j_3, \bar{j}_3)\delta_{\bar{j}_3, j_3+\frac{1}{2}}, \\
\langle \bar{j}^d_4 || a^\dagger[4] || j^d_4 \rangle = \langle \bar{j}^a_2 || a^\dagger[2] || j^a_2 \rangle = \Pi(j_4, \bar{j}_4)\delta_{\bar{j}_4, j_4+\frac{1}{2}}
\end{aligned}$$

Using the U(1) identifications and (4.12), (4.13), (4.16), (4.17) and (4.19) and merging all these equations carefully, we get:

$$\langle \bar{j}_{abcd} | \mathrm{Tr} U_{abcd} | j_{abcd} \rangle = -\delta_{j^a_3, \bar{j}^a_3}\delta_{j^a_4, \bar{j}^a_4}\delta_{j^a_{12}, \bar{j}^a_{12}}\delta_{j^b_1, \bar{j}^b_1}\delta_{j^b_4, \bar{j}^b_4}\delta_{j^c_1, \bar{j}^c_1}\delta_{j^c_2, \bar{j}^c_2}\delta_{j^c_{12}, \bar{j}^c_{12}}\delta_{j^d_2, \bar{j}^d_2}\delta_{j^d_3, \bar{j}^d_3}$$

$$(-1)^{j^a_{12}+j^c_{12}}(-1)^{j^b_1+j^b_4+j^d_2+j^d_3}\bar{\Pi}(j_1\bar{j}_1)\bar{\Pi}(j_2\bar{j}_2)\bar{\Pi}(j_3\bar{j}_3)\bar{\Pi}(j_4\bar{j}_4)\bar{\Pi}(j^b_{12}\bar{j}^b_{12})\bar{\Pi}(j^d_{12}\bar{j}^d_{12})$$

$$\begin{Bmatrix} j_1 & \bar{j}_1 & \frac{1}{2} \\ \bar{j}_4 & j_4 & j^a_{12} \end{Bmatrix}\begin{Bmatrix} j^b_{12} & \bar{j}^b_{12} & \frac{1}{2} \\ \bar{j}_1 & j_1 & j^b_4 \end{Bmatrix}\begin{Bmatrix} j^b_{12} & \bar{j}^b_{12} & \frac{1}{2} \\ \bar{j}_2 & j_2 & j^b_1 \end{Bmatrix}$$

$$\begin{Bmatrix} j_3 & \bar{j}_3 & \frac{1}{2} \\ \bar{j}_2 & j_2 & j^c_{12} \end{Bmatrix}\begin{Bmatrix} j^d_{12} & \bar{j}^d_{12} & \frac{1}{2} \\ \bar{j}_3 & j_3 & j^d_2 \end{Bmatrix}\begin{Bmatrix} j^d_{12} & \bar{j}^d_{12} & \frac{1}{2} \\ \bar{j}_4 & j_4 & j^d_3 \end{Bmatrix} \tag{4.20}$$

In (4.20), $\bar{\Pi}(a,b) \equiv (-1)^{a+b+\frac{1}{2}}\Pi(a,b)$. Note that $\bar{\Pi}(a,b)$ are symmetric $\bar{\Pi}(a,b) = \bar{\Pi}(b,a)$ and real. We have ignored the 16 δ functions $\prod_{i=1}^{4}\left(\delta_{\bar{j}_i, j_i+\frac{1}{2}} + \delta_{\bar{j}_i, j_i-\frac{1}{2}}\right)$ coming from the reduced matrix elements in (4.8) as they are already contained in the six $6j$ symbols in (4.20).

The final result can be written in a more compact form as:

$$\begin{aligned}
\langle \bar{j}_{abcd} | \mathrm{Tr} U_{abcd} | j_{abcd} \rangle \;=\; & \mathcal{M}_{abcd}\begin{Bmatrix} j_1 & \bar{j}_1 & \frac{1}{2} \\ \bar{j}_4 & j_4 & j^a_{12} \end{Bmatrix}\begin{Bmatrix} j^b_{12} & \bar{j}^b_{12} & \frac{1}{2} \\ \bar{j}_1 & j_1 & j^b_4 \end{Bmatrix}\begin{Bmatrix} j^b_{12} & \bar{j}^b_{12} & \frac{1}{2} \\ \bar{j}_2 & j_2 & j^b_1 \end{Bmatrix} \\
& \begin{Bmatrix} j_3 & \bar{j}_3 & \frac{1}{2} \\ \bar{j}_2 & j_2 & j^c_{12} \end{Bmatrix}\begin{Bmatrix} j^d_{12} & \bar{j}^d_{12} & \frac{1}{2} \\ \bar{j}_3 & j_3 & j^d_2 \end{Bmatrix}\begin{Bmatrix} j^d_{12} & \bar{j}^d_{12} & \frac{1}{2} \\ \bar{j}_4 & j_4 & j^d_3 \end{Bmatrix}.
\end{aligned} \tag{4.21}$$

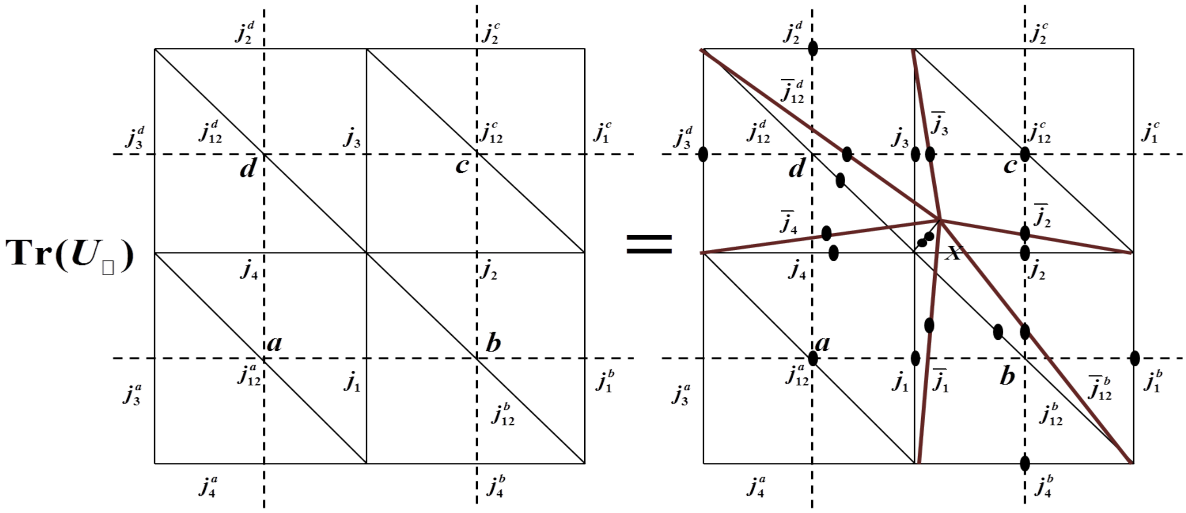

Figure 4.4: From loop kinematics to loop dynamics. (a) The angular momenta satisfying SU(2) Gauss law on the dual lattice, (b) The matrix elements $\langle \bar{j}_{abcd}|\mathrm{Tr}U_{abcd}|j_{abcd}\rangle$ with $X = \frac{1}{2}$ (also see [57]). The six tetrahedron are the six $6j$ symbols in (4.21). The unchanged j lines represent the delta function and a $\bullet$ over a j line represents the factor $(-1)^j\Pi(j)$ leading to $\mathcal{M}_{abcd}$ in (4.21).

In (4.21), $\mathcal{M}_{abcd} \equiv D_{abcd}N_{abcd}P_{abcd}$ where:

$$\begin{aligned}
D_{abcd} &= \delta_{j_3^a,\bar{j}_3^a}\delta_{j_4^a,\bar{j}_4^a}\delta_{j_{12}^a,\bar{j}_{12}^a}\delta_{j_1^b,\bar{j}_1^b}\delta_{j_4^b,\bar{j}_4^b}\delta_{j_1^c,\bar{j}_1^c}\delta_{j_2^c,\bar{j}_2^c}\delta_{j_{12}^c,\bar{j}_{12}^c}\delta_{j_2^d,\bar{j}_2^d}\delta_{j_3^d,\bar{j}_3^d}, \\
N_{abcd} &= \Pi\left(j_1,\bar{j}_1,j_2,\bar{j}_2,j_3,\bar{j}_3,j_4,\bar{j}_4,j_{12}^b,\bar{j}_{12}^b,j_{12}^d,\bar{j}_{12}^d\right) \\
P_{abcd} &= -(-1)^{j_1+j_2+j_1^b+j_4^b}(-1)^{j_3+j_4+j_3^d+j_2^d}\triangle(\bar{j}_1,\bar{j}_4,j_{12}^a)\triangle(\bar{j}_2,\bar{j}_3,j_{12}^c) \\
&\qquad \triangle(\bar{j}_{12}^b,j_{12}^b,\frac{1}{2})\triangle(\bar{j}_{12}^d,j_{12}^d,\frac{1}{2}).
\end{aligned} \tag{4.22}$$

In (4.22), D_{abcd} describes the trivial δ functions over the angular momenta which do not change under the action of the plaquette operator (4.2), N_{abcd} and P_{abcd} give the corresponding numerical and the phase factors respectively. The multiplicity factors are: $\Pi(x,y,...) \equiv \sqrt{(2x+1)(2y+1)...}$ and $\triangle(x,y,z)$ represent the phase factors associated with a triangle with sides x, y, z: $\triangle(x,y,z) \equiv (-1)^{x+y+z} \Rightarrow \triangle(x,y,z) = \pm 1$. The matrix elements (4.21) describe the dynamics in the loop basis (3.28) and can be geometrically represented by the Figure (4.4b). This dynamics contains three physical discrete angular momentum loop co-ordinates numbers per lattice site. The matrix elements (4.21) have been obtained[3] in the context of dual description [57] of $(2+1)$ dimension lattice gauge theory in terms of triangulated surfaces. It is easy to see that the matrix elements in (4.21) are symmetric: the 6j symbols satisfy $\left\{\begin{array}{ccc} x & \bar{x} & u \\ \bar{y} & y & v \end{array}\right\} = \left\{\begin{array}{ccc} \bar{x} & x & u \\ y & \bar{y} & v \end{array}\right\}$ and the factors D_{abcd}, N_{abcd} and P_{abcd} are individually symmetric under $j_{abcd} \leftrightarrow \bar{j}_{abcd}$. The matrix elements in (4.21) are also real as $\text{Tr}U_{\text{plaquette}}$ is a Hermitian operator. This reality can again be easily seen as the 6j symbols, $D_{abcd}, N_{abcd}, \triangle(abc)$ are themselves real. The remaining two phase factors $(-1)^{j_1+j_2+j_1^b+j_4^b}$ and $(-1)^{j_3+j_4+j_2^d+j_3^d}$ in P_{abcd} are real because (j_1, j_2, j_1^b, j_4^b) and (j_3, j_4, j_3^d, j_2^d) are the coordinates of the loop states at **b** and **d** respectively. Therefore, (3.16) implies: $j_1 + j_2 + j_1^b + j_4^b = \text{Integer},\ j_3 + j_4 + j_2^d + j_3^d = \text{Integer} \Rightarrow P_{abcd} = \pm 1$. We also cross check the d = 2 result (4.21). As the six 6j symbols and the δ functions in D_{abcd} are geometrical in origin, we only need to check the numerical and the phase factors $N_{abcd},\ P_{abcd}$ respectively. For this purpose, we replace $\text{Tr}U_{abcd}$ by the identity operator $\mathcal{I}$. Now we only have to replace $\frac{1}{2}$ in each of the six 6j symbols in (4.21) and in P_{abcd} in (4.22) by 0. Using the value $\left\{\begin{array}{ccc} a & \bar{a} & 0 \\ \bar{b} & b & d \end{array}\right\} = (-1)^{a+b+d}\delta_{a,\bar{a}}\delta_{b,\bar{b}}\left(\Pi(a,b)\right)^{-1}$, we get:

$$\left\{\begin{array}{ccc} j_1 & \bar{j}_1 & 0 \\ \bar{j}_4 & j_4 & j_{12}^a \end{array}\right\}\left\{\begin{array}{ccc} j_{12}^b & \bar{j}_{12}^b & 0 \\ \bar{j}_1 & j_1 & j_4^b \end{array}\right\}\left\{\begin{array}{ccc} ij_{12}^b & \bar{j}_{12}^b & 0 \\ \bar{j}_2 & j_2 & j_1^b \end{array}\right\}$$
$$\left\{\begin{array}{ccc} j_3 & \bar{j}_3 & 0 \\ \bar{j}_2 & j_2 & j_{12}^c \end{array}\right\}\left\{\begin{array}{ccc} j_{12}^d & \bar{j}_{12}^d & 0 \\ \bar{j}_3 & j_3 & j_2^d \end{array}\right\}\left\{\begin{array}{ccc} j_{12}^d & \bar{j}_{12}^d & 0 \\ \bar{j}_4 & j_4 & j_3^d \end{array}\right\}$$
$$= \frac{\delta_{j_1,\bar{j}_1}\delta_{j_2,\bar{j}_2}\delta_{j_3,\bar{j}_3}\delta_{j_4,\bar{j}_4}\delta_{j_{12}^b,\bar{j}_{12}^b}\delta_{j_{12}^d,\bar{j}_{12}^d}}{N_{abcd}\ P_{abcd}}, \tag{4.23}$$

Geometrically, the equation (4.23) corresponds to putting $X = 0$ in the Figure (4.4b). It implies $\langle \bar{j}_{abcd}|\mathcal{I}|j_{abcd}\rangle = \delta_{\bar{j}_{abcd},j_{abcd}}$ confirming the numerical and the phase factors in (4.21). We now write (4.21) in a more compact form which can be directly generalized to higher dimension. Henceforth, we ignore D_{abcd} representing trivial δ functions in (4.21).

[3]Our phase factors in (4.22) are different resulting in real and symmetric matrix $\langle \bar{j}_{abcd}|\text{Tr}U_{abcd}|j_{abcd}\rangle$ in (4.21).

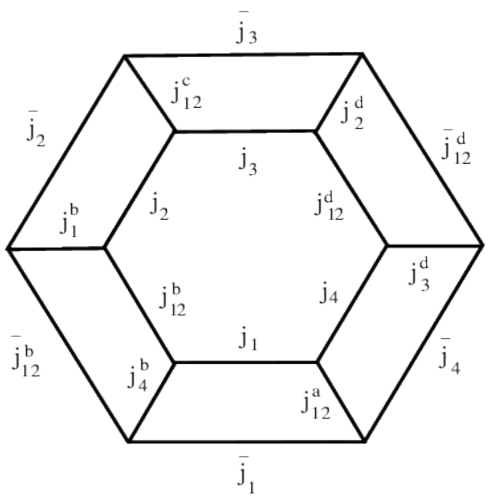

Figure 4.5: The $18j$ ribbon diagram representing exact SU(2) loop dynamics without any spurious gauge or loop degrees of freedom in d=2. The interior (exterior) edge carries the initial (final) angular momenta and the six bridges carry the angular momenta which are invariant under the action of $\mathrm{T}rU_{\Box}$. The six bridges along with the respective four angular momenta attached represent the six $6j$ symbols appearing in (4.25) with $\bar{j}_i = j_i \pm \frac{1}{2}$.

We write:

$$\langle \bar{j}_{abcd}|\mathrm{Tr}U_{abcd}|j_{abcd}\rangle = N_{abcd}\sum_x (2x+1)(-1)^{r+2x}\begin{Bmatrix} j_1 & \bar{j}_1 & x \\ \bar{j}_4 & j_4 & j^a_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{j}_4 & j_4 & x \\ j^d_{12} & \bar{j}^d_{12} & j^d_3 \end{Bmatrix}\begin{Bmatrix} j^d_{12} & \bar{j}^d_{12} & x \\ \bar{j}_3 & j_3 & j^d_2 \end{Bmatrix}\begin{Bmatrix} \bar{j}_3 & j_3 & x \\ j_2 & \bar{j}_2 & j^c_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} j_2 & \bar{j}_2 & x \\ \bar{j}^b_{12} & j^b_{12} & j^b_1 \end{Bmatrix}\begin{Bmatrix} \bar{j}^b_{12} & j^b_{12} & x \\ j_1 & \bar{j}_1 & j^b_4 \end{Bmatrix}\prod_{i=1}^{4}\left(\delta_{\bar{j}_i,j_i+\frac{1}{2}} + \delta_{\bar{j}_i,j_i-\frac{1}{2}}\right) \tag{4.24}$$

$$= N_{abcd}\underbrace{\begin{bmatrix} j_1 & & j_4 & & j^d_{12} & & j_3 & & j_2 & & j^b_{12} & \\ & j^a_{12} & & j^d_3 & & j^d_2 & & j^c_{12} & & j^b_1 & & j^b_4 \\ \bar{j}_1 & & \bar{j}_4 & & \bar{j}^d_{12} & & \bar{j}_3 & & \bar{j}_2 & & \bar{j}^b_{12} & \end{bmatrix}}_{\text{18j coefficient of the second kind}}\prod_{i=1}^{4}\left(\delta_{\bar{j}_i,j_i+\frac{1}{2}} + \delta_{\bar{j}_i,j_i-\frac{1}{2}}\right)$$

The 18j symbols in (4.25) are shown in (4.5). Note that $P_{abcd}\,(= (-1)^{r+1})$ in (4.22) is precisely the phase factor needed to define 18j symbol [39] in (4.25). Further, the 12 triangular constraints in (4.25) at the 12 vertices of the ribbon diagram in Figure (4.5) are already solved in terms of the linking numbers. Therefore it is only the value of the $3nj = 18j$ (n = 6) symbol which is important. The form (4.25) also makes reality and symmetry of $\langle \bar{j}_{abcd}|\mathrm{TrU_{abcd}}|j_{abcd}\rangle$ manifest as 3nj symbols of second kind are real and symmetric:

$$\begin{bmatrix} j_1 & & j_4 & & j^d_{12} & & j_3 & & j_2 & & j^b_{12} & \\ & j^a_{12} & & j^d_3 & & j^d_2 & & j^c_{12} & & j^b_1 & & j^b_4 \\ \bar{j}_1 & & \bar{j}_4 & & \bar{j}^d_{12} & & \bar{j}_3 & & \bar{j}_2 & & \bar{j}^b_{12} & \end{bmatrix}$$

$$= \begin{bmatrix} \bar{j}_1 & & \bar{j}_4 & & \bar{j}^d_{12} & & \bar{j}_3 & & \bar{j}_2 & & \bar{j}^b_{12} & \\ & j^a_{12} & & j^d_3 & & j^d_2 & & j^c_{12} & & j^b_1 & & j^b_4 \\ j_1 & & j_4 & & j^d_{12} & & j_3 & & j_2 & & j^b_{12} & \end{bmatrix}$$

Before going to arbitrary dimension, we make the following simple observation. Let $\Delta N_x,\ x = a, b, c, d$ denote the number of angular momenta appearing in the loop states $|j_{abcd}\rangle$ in (4.3) which change under the action of the plaquette operator $\mathrm{T}rU_{abcd}$ at lattice site x. In the present, $d = 2$, case:

$$\Delta N_a = 2,\ \ (j^a_1, j^a_2)\,;\ \ \Delta N_b = 3,\ \ \left(j^b_2, j^b_3, j^b_{12}\right);\ \ \Delta N_c = 2,\ \ (j^c_3, j^c_4)\,;\ \ \Delta N_d = 3,\ \ \left(j^d_1, j^d_4, j^d_{12}\right).$$

The $U(1)$ identification (4.4) implies double counting on each of the 4 links of the plaquette *abcd*. Therefore, the number of angular momenta which change under the action of the plaquette in the (12) plane: $\Delta N(12) = \Delta N_a + \Delta N_b + \Delta N_c + \Delta N_d - 4 = 10 - 4 = 6 = n$. This analysis will be useful to generalize the loop dynamics to arbitrary dimensions below.

4.3 d dimension

It is clear from the previous section that the loop dynamics in d dimension is also given in terms of $3nj$ symbols. However, in arbitrary d dimension, n will depend on the orientation of the plaquette. We will now compute n. We consider the plaquette **abcd** in the $(I, K), I < K$ plane as shown in Figure (4.6). Like in $d = 2$, we consider the loop states over the plaquette *abcd*:

$$|j_{abcd}\rangle \equiv |LS\rangle_a \otimes |LS\rangle_b \otimes |LS\rangle_c \otimes |LS\rangle_d \tag{4.25}$$

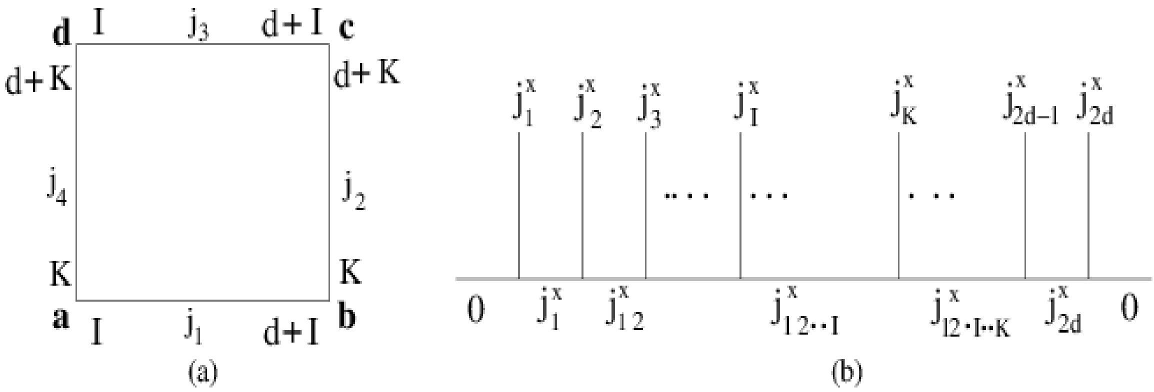

Figure 4.6: (a) The plaquette **abcd** in the (I, K) plane in d dimension. We choose $I < K$, $1 \leq I < d$ and $1 < K \leq d$, (b) The angular momentum addition scheme at site x (=**a**,**b**,**c**,**d**). Note that j_1^x and j_{2d}^x appear twice in the scheme. The δ functions are subtracted in (4.26) to avoid this double counting.

where $|LS\rangle_{x=a,b,c,d} = |j_1^x, j_2^x, j_{12}^x, j_3^x, j_{123}^x, ..., j_I^x, j_{12...I}^x, .., j_K^x, j_{12..I..K}^x, ..., j_{2d-1}^x, j_{12..(2d-1)}^x (= j_{2d}^x), j_{2d}^x, 0\rangle$. We now have to count the the number of angular momenta in (4.25) which change under the action of the plaquette operator $\mathrm{T}rU_{abcd}$ in the (IK) plane. With the choice $I < K$ $(1 \leq I < d, 1 < K \leq d)$, we have:

$$\Delta N_a = 2 + (K - I) - \delta_{I,1}, \qquad \Delta N_b = 2 + (d + I) - K,$$
$$\Delta N_c = 2 + (K - I) - \delta_{K,d}, \qquad \Delta N_d = 2 + (d + K) - I - \delta_{I,1} - \delta_{K,d} \qquad (4.26)$$

This implies:

$$\Delta N(IK) = \Delta N_a + \Delta N_b + \Delta N_c + \Delta N_d - 4 = 2\left[2 + d + (K - I) - \delta_{I,1} - \delta_{K,d}\right] = n(IK) \qquad (4.27)$$

Like in d =2 case, we have subtracted 4 in (4.27) because of U(1) gauge invariance. Note that for d=2, $\Delta N(12) = 6$ and for d=3, $\Delta N(12) = \Delta N(13) = \Delta N(23) = 10$. The d = 3 loop dynamics is explicitly shown in Figure (4.7) where we have used the notations from Figure (4.6a), i.e.:

$$(I = 1, K = 2) = (12) \text{ plane} : j_1^a = j_4^b = j_1,\ j_2^b = j_5^c = j_2,\ j_4^c = j_1^d = j_3, j_5^d = j_2^a = j_4,$$
$$(I = 1, K = 3) = (13) \text{ plane} : j_1^a = j_4^b = j_1,\ j_3^b = j_6^c = j_2,\ j_4^c = j_1^d = j_3, j_6^d = j_3^a = j_4,$$
$$(I = 2, K = 3) = (23) \text{ plane} : j_2^a = j_5^b = j_1,\ j_3^b = j_6^c = j_2,\ j_5^c = j_2^d = j_3, j_6^d = j_3^a = j_4.$$

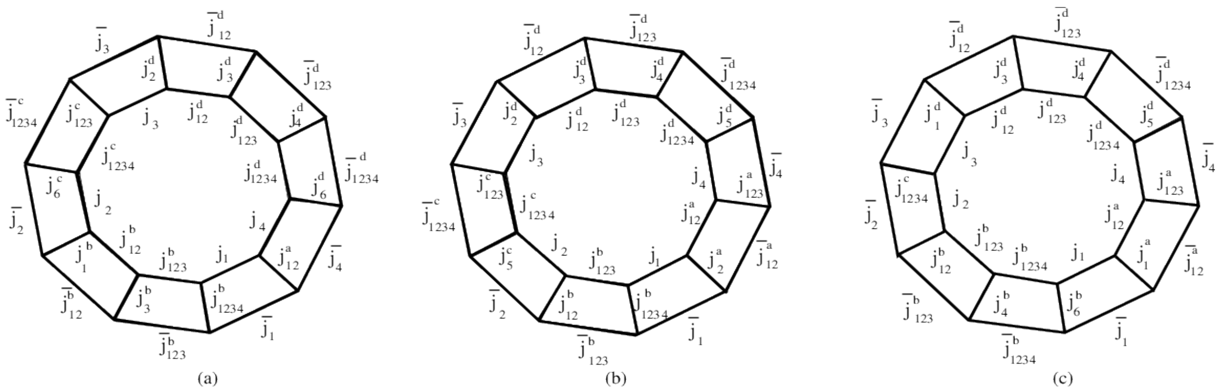

Figure 4.7: The $30j$ ribbon diagrams representing exact SU(2) loop dynamics without any spurious gauge or loop degrees of freedom in d=3: a) (12) plane, b) (13) plane, c) (23) plane. The angular momenta j_1, j_2, j_3 and j_4 are as shown in Figure (4.6a). The initial (inner) and final (outer) angular momenta differ by $\frac{1}{2}$.

It is clear from (4.27) that in higher ($d > 3$) dimension $\Delta N(IK)$ depends on the orientation of the plaquette. The corresponding $3n(I,K)j$ symbol describing the loop dynamics in the above angular momentum addition scheme can be easily written down.

4.4 Summary and discussion

In this chapter, we have calculated the dynamics of the theory within the orthonormal loop basis defined around a plaquette. The matrix element of $\text{Tr}U_{plaquette}$ within the loop basis comes as 3nj symbols. This result can further be used to calculate the spectrum of the full Hamiltonian.

Having established the advantages of prepotential formulation with the illustration via SU(2) group over the last three chapters, we now generalize this approach to SU(3) lattice gauge theory, as SU(3) is the gauge group associated with quantum chromodynamics. We give this generalization in the next chapter. We further generalize this novel prepotential formulation to arbitrary SU(N) gauge group in chapter 6. It is also possible to exploit prepotential approach in order to solve the lattice Schrödinger equation and calculate the spectrum of the theory. We have also made progress in this direction. In chapter 7 we explicitly solve the schroedinger equation for a small lattice analytically.

Chapter 5

Prepotential Formulation of SU(3) Lattice Gauge Theory

We have discussed in previous chapter the new prepotential variables for SU(2) lattice gauge theory. We have also seen how the loop state dynamics gets simplified in terms of these prepotential operators compared to the same in terms of original Kogut-Susskind variables for SU(2). However, at the end, we have to work with the gauge group SU(3) to solve actual problems of QCD. As in the case of SU(2), the loop formulation of SU(3) lattice gauge theory also suffers from the problem of non locality and proliferation of loop states. In the last chapter the SU(2) prepotentials were shown to be useful in formulating the theory entirely in terms of physical variables or loops. In this present chapter we generalize this approach to SU(3) gauge group by defining and developing the prepotential formulation to SU(3) Lattice gauge theory.

As we know earlier prepotentials are basically harmonic oscillators belonging to the fundmental representations of the gauge group. In the context of SU(2) group theory, the Schwinger bosons, each carrying basic (half) unit of spin angular momentum flux, provide an explicit and simple realization of the angular momentum algebra as well as all it's representations [33]. In particular, the Hilbert space created by the two Schwinger oscillators is isomorphic to the space of SU(2) irreps. Thus the Schwinger boson representation of SU(2) group is simple, economical as well as complete. However, all these features are lost when we generalize the Schwinger boson construction to SU(N) with $N \geq 3$. The origin of these problems is the existence of certain SU(N) invariants which can be constructed for $N \geq 3$. Any two states which differ by the overall presence of such an invariants will transform in the same way under SU(N) gauge transformation. This leads to the problem of multiplicity which in turn makes the formulation of SU(N) ($N \geq 3$) much more involved compared to SU(2). To understand this fact let us consider the Schwinger boson representation for SU(3) explicitly. Being rank two group two fundamental irrep of SU(3), namely the $a^\dagger_\alpha \in 3$ and $b^{\dagger\alpha} \in 3^*$ are necessary to construct all possible irreps of SU(3). Now we can see that $a^\dagger \cdot b^\dagger$ as well as $a \cdot b$ are SU(3) invariant. Hence for any irrep $|R\rangle$ of SU(3) should be identified with $|R, \rho\rangle$ for any ρ, where,

$$|R, \rho\rangle \equiv \underbrace{(a^\dagger \cdot b^\dagger)^\rho}_{\text{SU(3) singlet}} |R\rangle \tag{5.1}$$

for $\rho = 0, 1, 2, \ldots, \infty$. Note that, this degeneracy problem did not exist for SU(2) as there is only one ($2 \equiv 2^*$) fundamental representation of SU(2). However this group gauge degeneracy can be solved by solving the group theoretic Gauss law constraint $a \cdot b \simeq 0$ [34]. The degree of degeneracy increases with N for gauge group SU(N) leading to more and more complicated Schwinger boson representation of SU(N). These issues have also been extensively addressed in the past [19,30,40,41]. In the context of SU(3) Schwinger boson analysis, a systematic group theoretic procedure based on noncompact group Sp(2,R) is given in [40] to label the multiplicity of SU(3). In this work, exploiting this Sp(2,R) labeling in [40], we define irreducible SU(3) Schwinger bosons in terms of which construction of SU(3) irreducible representations are as simple as SU(2). Further like in SU(2) case, the representations in terms of irreducible Schwinger bosons are multiplicity free.

We identify this new set of irreducible Schwinger bosons for SU(3) as the irreducible prepotential operators to formulate the SU(3) lattice gauge theory entirely in terms of these irredicible prepotentials. In this chapter we show how the irreducible prepotentials are constructed from the group theoretic properties. We also construct the conventional variables, the electric field and link operator in terms of prepotentials. We also show that the prepotentials being located at the end of each link, enables one to construct physical states locally at each site. The Mandelstam constraints can also be cast in its local form.

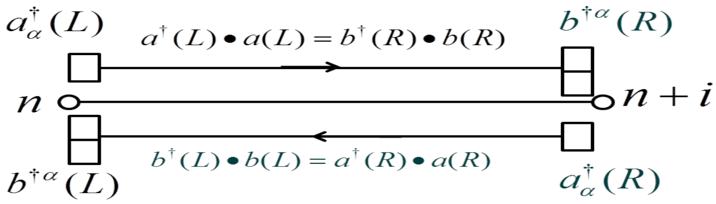

Figure 5.1: The SU(3) prepotentials and the two $U(1) \otimes U(1)$ oriented abelian flux lines along a link l in SU(3) lattice gauge theory. The directions of abelian flux lines are chosen from quarks $a^\dagger \in 3$ prepotentials to anti-quark $b^\dagger \in 3^*$ prepotentials transforming like triplet and ant-itriplet respectively.

5.1 Prepotentials in SU(3) lattice gauge theory: Definition and Construction

Like in SU(2), the SU(3) prepotentials are defined through the left and right electric fields in SU(3) lattice gauge theory. As stated earlier, being a group of rank two, SU(3) has two fundamental representations 3 (triplet) and 3^* (anti-triplet) which are independent. Hence we associate two independent harmonic oscillator prepotential triplets:

$$a^\dagger_\alpha(l, L), \qquad b^{\dagger\alpha}(l, L), \qquad \alpha = 1, 2, 3$$

to the left end and

$$a^\dagger_\alpha(l; R), \qquad b^{\dagger\alpha}(l; R), \qquad \alpha = 1, 2, 3$$

to the right end of the link $l \equiv (n, i)$. Now there are 12 prepotential operators associated with every link. These assignments are shown in Figure 5.1. Under SU(3) gauge transformation in a d dimensional spatial lattice, the $2d$ $a^\dagger s$ and 2d $b^\dagger s$ on the 2d links emanating from the lattice site n transform as quarks (3) and anti-quarks (3^*) respectively. The SU(3) electric fields are related to the prepotentials as:

$$\begin{aligned} \text{Left electric fields:} \qquad E^{\rm a}_L(l) &= \left(a^\dagger(l, L)\frac{\lambda^{\rm a}}{2}a(l, L) - b(l, L)\frac{\lambda^{\rm a}}{2}b^\dagger(l, L)\right) \\ \text{Right electric fields:} \qquad E^{\rm a}_R(l) &= \left(a^\dagger(l, R)\frac{\lambda^{\rm a}}{2}a(l, R) - b(l, R)\frac{\lambda^{\rm a}}{2}b^\dagger(l, R)\right) \end{aligned} \tag{5.2}$$

In (5.2), we have used Schwinger boson construction of SU(3) Lie algebra [45, 59]. Note that the second part of the each of these definition is $-b\frac{\lambda^{\rm a}}{2}b^\dagger$ instead of $b^\dagger\frac{\tilde{\lambda}^{\rm a}}{2}b$ where, $\tilde{\lambda}$

is the dual representation of λ as we have used the relation $\tilde{\lambda} = -\lambda^T$. The electric field generators in (5.2) generate $SU_L(3) \otimes SU_R(3)$ gauge transformations on every link. The Electric field operators defined in (5.2) also satisfy the rigit rotator constraint (2.6), i,e

$$\sum_{a=1}^{8} E_L^{\mathrm{a}}(l) E_L^{\mathrm{a}}(l) = \sum_{a=1}^{8} E_R^{\mathrm{a}}(l) E_R^{\mathrm{a}}(l). \tag{5.3}$$

The prepotential triplets satisfy the standard harmonic oscillator commutation relations:

$$\begin{aligned} \left[a^\alpha(l,s), a^\dagger_\beta(l,s')\right] &= \delta^\alpha_\beta \delta_{s,s'}\ , & \left[b_\alpha(l,s), b^{\dagger\beta}(l,s')\right] = \delta^\beta_\alpha \delta_{s,s'} \\ \left[a^\alpha(l,s), a^\beta(l,s')\right] &= 0\ , & \left[b_\alpha(l,s), b_\beta(l,s')\right] = 0, \quad s,s' = L,R. \end{aligned} \tag{5.4}$$

As all the electric fields in (5.2) involve both creation and annihilation operators, the number operators in (5.5) commute with all the electric fields in (5.2). Therefore, the two SU(3) Casimirs on each side of the link l are:

$$\begin{aligned} \hat{N}(l,L) &= a^\dagger(l,L)\cdot a(l,L), & \hat{M}(l,R) &= b^\dagger(l,R)\cdot b(l,R), \\ \hat{M}(l,L) &= b^\dagger(l,L)\cdot b(l,L), & \hat{N}(l,R) &= a^\dagger(l,R)\cdot a(l,R). \end{aligned} \tag{5.5}$$

The eigenvalues of $\hat{N}(L), \hat{M}(L)$ and $\hat{N}(R), \hat{M}(R)$ will be denoted by n_L, m_L and n_R, m_R respectively. We can characterize all the SU(3) irreducible representations on a link by $(n_L, m_L) \otimes (n_R, m_R)$. Under the $SU(3)$ gauge transformations, the prepotentials on the left and right side of a link l transform as:

$$\begin{aligned} a^\dagger_\alpha(l,L) &\to a^\dagger_\beta(l,L)\left(\Lambda^\dagger_L\right)^\beta{}_\alpha, & a^\dagger_\alpha(l,R) &\to a^\dagger_\beta(l,R)\left(\Lambda^\dagger_R\right)^\beta{}_\alpha \\ b^{\dagger\alpha}(l,L) &\to \left(\Lambda_L\right)^\alpha{}_\beta b^{\dagger\beta}(l,L), & b^{\dagger\alpha}(l,R) &\to \left(\Lambda_R\right)^\alpha{}_\beta b^{\dagger\beta}(l,R) \end{aligned} \tag{5.6}$$

The above transformations imply that under SU(3) gauge transformations $a^\dagger_\alpha(l,L), a^\dagger_\alpha(l,R)$ transform like quarks and $b^{\dagger\alpha}(l,L), b^{\dagger\alpha}(l,R)$ transform like anti-quarks at the left and the right end of the link l respectively. Therefore, we call $a, a^\dagger$ and $b, b^\dagger$ on various links as quark and anti quark prepotentials respectively. Note that, under SU(3) gauge transformation Kogut-Susskind link operator transforms as,

$$U_\alpha{}^\beta(l) \to \Lambda_\alpha{}^\gamma(l,L) U_\gamma{}^\delta(l) \Lambda^\dagger_\delta(l,R)^\beta. \tag{5.7}$$

We will see in the next sections that the prepotentials in (5.2) enable us to disentangle the left and right SU(3) gauge flux formation.

5.2 The additional $U(1) \otimes U(1)$ gauge invariance

Like in SU(2) case , the defining equations of SU(3) prepotentials (5.2) on link l are invariant under the following $U(1) \otimes U(1) \otimes U(1) \otimes U(1)$ abelian gauge transformations:

$$\begin{aligned} a^\dagger_\alpha(l,L) &\to e^{i\theta(l,L)} a^\dagger_\alpha(l,L), & b^{\dagger\alpha}(l,R) &\to e^{-i\theta(l,R)} b^{\dagger\alpha}(l,R), \\ b^{\dagger\alpha}(l,L) &\to e^{i\phi(l,L)} b^{\dagger\alpha}(l,L), & a^\dagger_\alpha(l,R) &\to e^{-i\phi(l,R)} a^\dagger_\alpha(l,R) \end{aligned} \quad (5.8)$$

In (5.8), the abelian gauge angles $\theta(l,s)$ and $\phi(l,s)$ with $s = L, R$ are defined on the left and right sides of every link. Again like in SU(2) case, the Hilbert space of lattice gauge theory is built by applying the link operators on the vacuum state:

$$U^{\alpha_1}{}_{\beta_1}(l)\; U^{\alpha_2}{}_{\beta_2}(l) \cdots U^{\alpha_n}{}_{\beta_n}(l)|0\rangle \quad (5.9)$$

and then symmetrizing/anti-symmetrizing α indices according to a certain Young tableau. However, this symmetrizing/anti-symmetrizing the left $\alpha(\in 3)$ indices automatically induces the same symmetries/anti-symmetries on the right $\beta(\in 3^*)$ indices. This implies that the left and right representations are always conjugate to each other[1], i.e:

$$\hat{N}(l,L) \simeq \hat{M}(l,R), \qquad \hat{M}(l,L) \simeq \hat{N}(l,R), \quad (5.10)$$

where $\simeq$ denotes that it is true acting on the states created by the action of link operators on vacuum. This implies that within the Hilbert space created by the link operators in (5.9), $\theta(l,L) = \theta(l,R)$ and $\phi(l,L) = \phi(l,R)$ is true on every link l. A trivial example is the state $|^{\beta}_{\alpha}\rangle \equiv U_\alpha{}^\beta(l)|0\rangle$ which transform as 3-plet on the left and 3^*-plet on the right. The next irreducible state $|^{\beta_1\beta_2}_{\alpha_1\alpha_2}\rangle \equiv \big(U_{\alpha_1}{}^{\beta_1}(l)U_{\alpha_2}{}^{\beta_2}(l) + U_{\alpha_2}{}^{\beta_1}(l)U_{\alpha_1}{}^{\beta_2}(l)\big)$ transforms as 6 at left and 6^* at right of the link l. Note that, the normalization of these states are nontrivial as it carries all the symmetrization as well as antisymmetrization of the α and β indices. These same states (according to the transformation properties) are realized in terms of prepotential operators as:

$$|^{\beta}_{\alpha}\rangle \equiv a^\dagger_\alpha(l,L) b^{\dagger\beta}(l,R)|0\rangle \;\;,\;\; |^{\beta_1\beta_2}_{\alpha_1\alpha_2}\rangle \equiv a^\dagger_{\alpha_1}(l,L) a^\dagger_{\alpha_2}(l,L) b^{\dagger\beta_1}(l,R) b^{\dagger\beta_2}(l,R)|0\rangle. \quad (5.11)$$

It is evident in (5.11) that the (5.10) is true. Therefore, besides SU(3) gauge invariance at different lattice sites, the prepotential formulation has additional abelian $U(1) \otimes U(1)$

[1] We will analyze the consequences of $E^2_L(n,i) = E^2_R(n+i,i)$ in the next section.

gauge invariance (5.8) on every link. The abelian Gauss law constraints (5.10) imply oriented abelian fluxes for SU(3) unlike the unoriented ones for SU(2) as in this case there exist more than one (triplet and anti-triplet) type of fundamental representation attached to each end of the link. Hence it is necessary to choose a convention of the abelian Gauss law flow in order to express it clearly. We choose the directions of the abelian fluxes on links to be from quark to anti quark prepotentials. To maintain continuity of direction in a loop state the non-abelian fluxes are chosen in the opposite direction (i.e, from anti quark prepotentials to quark prepotentials). These conventions are clearly illustrated on a link in Figure 5.1 and Figure 5.2.

5.3 The SU(3) prepotential Hilbert space $\mathcal{H}_{\rm p}$ on a link

Likewise the SU(2) case, the Hilbert space of SU(3) prepotential operators $\mathcal{H}_p$ can be completely characterized by the following basis on every lattice link l (we supress the link index l as long as we consider only one link at a time):

$$\begin{aligned} |^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}\rangle_L \otimes |^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}\rangle_R &\equiv \hat{L}\,^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \otimes \hat{R}\,^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \\ &\equiv |p,q\rangle_L \otimes |q,p\rangle_R \end{aligned} \tag{5.12}$$

In (5.12),

$$|p,q\rangle_L = \hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \;\equiv\; a^\dagger_{\alpha_1}(L)\cdots a^\dagger_{\alpha_p}(L) b^{\dagger\beta_1}(L)\cdots b^{\dagger\beta_q}(L)|0\rangle_L, \tag{5.13}$$

and

$$|q,p\rangle_R = \hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \;\equiv\; a^\dagger_{\gamma_1}(R)\cdots a^\dagger_{\gamma_q}(R) b^{\dagger\delta_1}(R)\cdots b^{\dagger\delta_p}(R)|0\rangle_R \tag{5.14}$$

are the states created by the $SU_L(3)\otimes SU_R(3)\otimes U(1)\otimes U(1)$ flux creation operators on the left and right vacuum at respective ends of every link. We have used the $U(1)\otimes U(1)$ Gauss law constraints (5.10) in (5.12) with $n_L = m_R = p$ and $m_L = n_R = q$.

Note that unlike SU(2) flux creation operators (2.16) which were SU(2) irreducible, the flux operators in (5.12) which creates the states in prepotential Hilbert space are SU(3) reducible. To construct the SU(3) irreducible states at each end of the link the prepotentials need to be properly symmetrized and antisymmetrized as the SU(3) irrep must be symmetric in all upper as well as lower indices, and must be antisymmetric between upper and lower indices. Note that this symmetrization was inbuilt for SU(2) as that have only one type of indices which were already symmetric as all the Schwinger bosons commute amongst themselves. The same is true for symmetric or pure SU(3) irreps, like $(p,0)$ and $(0,p)$. The symmetric representations of SU(3) are created out of only one fundamental irrep, either 3 or 3^*. Therefore, as far as symmetric representations of SU(3) are concerned, each single (double) Young tableau box represents a Schwinger boson creation operator $a^\dagger \in 3$ $\left(b^\dagger \in 3^*\right)$. Hence the symmetric SU(3) irreps at both the ends of a link are :

$$\begin{aligned} |p,0\rangle_L &= \hat{L}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv a^\dagger_{\alpha_1}(L)\cdots a^\dagger_{\alpha_p}(L)|0\rangle_L, \\ |0,p\rangle_R &= \hat{R}^{\beta_1\beta_2\cdots\beta_p}|0\rangle_R \equiv b^{\dagger\beta_1}(R)\cdots b^{\dagger\beta_p}(R)|0\rangle_R \end{aligned} \tag{5.15}$$

In (5.15) the states created by the $SU_L(3)\otimes SU_R(3)\otimes U(1)\otimes U(1)$ flux creation operators on the left and right vacuum at respective ends of every link. Moreover this flux creation operators are symmetric as $a^\dagger$'s and $b^\dagger$'s commute among themselves. Hence the states $|p,0\rangle_L$ and $|0.q\rangle_R$ are indeed irreducible representation of SU(3) and are equivalent to a row of p number of single boxes and a row of p number of double boxes respectively.. This construction is simple and retain the simplicity of SU(2). However, this simplicity is lost when we consider mixed representations ($p\neq 0, q\neq 0$). As stated earlier, the states in (5.13) and (5.14) are SU(3) reducible. The irreducible representation states in (p,q) irrep. are given by [30]:

$$\begin{aligned} |\psi\rangle^{\alpha_1,\alpha_2,\ldots\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q} \equiv & \Big[O^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q} + L_1\sum_{l_1=1}^{p}\sum_{k_1=1}^{q}\delta^{\alpha_{l_1}}_{\beta_{k_1}}O^{\alpha_1\alpha_2..\alpha_{l_1-1}\alpha_{l_1+1}..\alpha_p}_{\beta_1\beta_2..\beta_{k_1-1}\beta_{k_1+1}\cdots\beta_q} \\ & +L_2\sum_{\binom{l_1,l_2}{=1}}^{p}\sum_{\binom{k_1,k_2}{=1}}^{q}\delta^{\alpha_{l_1}\alpha_{l_2}}_{\beta_{k_1}\beta_{k_2}}O^{\alpha_1..\alpha_{l_1-1}\alpha_{l_1+1}..\alpha_{l_2-1}\alpha_{l_2+1}..\alpha_p}_{\beta_1..\beta_{k_1-1}\beta_{k_1+1}..\beta_{k_2-1}\beta_{k_2+1}..\beta_q} \\ & +L_3\sum_{\binom{l_1,l_2}{l_3=1}}^{p}\sum_{\binom{k_1,k_2}{k_3=1}}^{q}\delta^{\alpha_{l_1}\alpha_{l_2}\alpha_{l_3}}_{\beta_{k_1}\beta_{k_2}\beta_{k_3}}O^{\alpha_1..\alpha_{l_1-1}\alpha_{l_1+1}..\alpha_{l_2-1}\alpha_{l_2+1}..\alpha_{l_3-1}\alpha_{l_3+1}..\alpha_p}_{\beta_1..\beta_{k_1-1}\beta_{k_1+1}..\beta_{k_2-1}\beta_{k_2+1}..\beta_{k_3-1}\beta_{k_3+1}\cdots\beta_q} \\ & \vdots \\ & +L_n\sum_{l_1..l_n=1}^{p}\sum_{k_1..k_n=1}^{q}\delta^{\alpha_{l_1}\alpha_{l_2}..\alpha_{l_n}}_{\beta_{k_1}\beta_{k_2}..\beta_{k_n}}O^{\alpha_1\alpha_2..\alpha_{l_1-1}\alpha_{l_1+1}..\alpha_{l_2-1}\alpha_{l_2+1}..\alpha_{l_n-1}\alpha_{l_n+1}..\alpha_p}_{\beta_1\beta_2..\beta_{k_1-1}\beta_{k_1+1}..\beta_{k_2-1}\beta_{k_2+1}..\beta_{k_n-1}\beta_{k_n+1}\cdots\beta_q}\Big]|0\rangle \end{aligned} \tag{5.16}$$

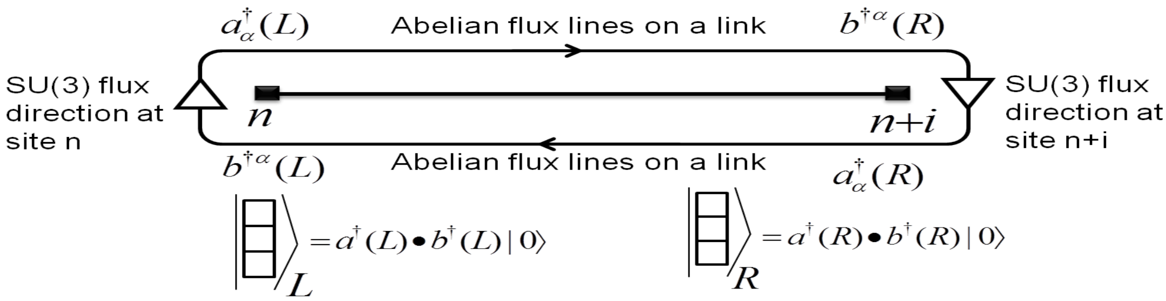

Figure 5.2: The graphical interpretation of the $SU(3)\otimes U(1)\otimes U(1)$ gauge invariant loop state (5.18) over a link (n,i) with $n_L = n_R = n = 1$. The “magnetic” Sp(2,R) quantum number ρ of this state is non zero ($\rho = 1$) and therefore such states can not be created by the link operators $U(n,i)$. Two types of arrows are used to differentiate abelian and non-abelian fluxes.

where $O \equiv \hat{L}(\hat{R})$ which are defined in (5.13) and (5.14). The limit in the sum $n = \min(p,q)$, $\delta^{\alpha_1\alpha_2..\alpha_r}_{\beta_1\beta_2...\beta_r} \equiv \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\ldots\delta^{\alpha_r}_{\beta_r}$ and all the sums in (5.16) are over different indices, i.e, $l_1 \neq l_2 ... \neq l_n$ and $k_1 \neq k_2 ... \neq k_n$. The coefficients L_r are given by [59]:

$$L_r \equiv \frac{(-1)^r\ (a^\dagger\cdot b^\dagger)^r}{(p+q+1)(p+q)(p+q-1)...(p+q+2-r)}\ . \tag{5.17}$$

The coefficients in (5.17) are chosen to make the state completely traceless in all possible pairs of upper and lower indices. Note that all the upper (lower) indices are completely symmetric by construction as the Schwinger bosons commute amongst themselves.

The non-triviality with the mixed irreps (i.e constructed out of two fundamental irreps) arises due to the existence of certain SU(3)invariant operators. To understand the problem clearly, let us consider the following SU(3) gauge invariant state as an example:

$$|\rho_L,\ \rho_R\rangle \equiv \Big(a^\dagger(L)\cdot b^\dagger(L)\Big)^{\rho_L}\Big(a^\dagger(R)\cdot b^\dagger(R)\Big)^{\rho_R}|0\rangle. \tag{5.18}$$

The states (5.18) are also invariant under $U(1)\otimes U(1)$ gauge transformations (5.8) if $\rho_L = \rho_R = \rho$ with $\rho = 0,1,2,...,\infty$. The state (5.18) with $\rho = 1$ is shown in Figure 5.2. The gauge invariant states (5.18) are linear combinations of states in (5.12):

$$|\rho\rangle \equiv |\rho_L = \rho,\ \rho_R = \rho\rangle = \sum_{\{\alpha_i\}_{i=1,..,\rho}} |^{\alpha_1\alpha_2\cdots\alpha_\rho}_{\alpha_1\alpha_2\cdots\alpha_\rho}\rangle_L \otimes \sum_{\{\beta_i\}_{i=1,..,\rho}} |^{\beta_1\beta_2\cdots\beta_\rho}_{\beta_1\beta_2\cdots\beta_\rho}\rangle_R. \tag{5.19}$$

However, the gauge invariant states in pure lattice gauge theories are the Wilson loop states residing around the plaquettes and *not on the links* . The only gauge invariant operator on a link l constructed out of the link operator is a constant operator $\{\mathrm{Tr}(U^\dagger(l)U(l)) = \mathrm{Tr}\ (U(l)U^\dagger(l)) = 3\}$. In other words, the infinite towers of gauge invariant states (5.18) on different links do not exist in the lattice gauge theory. In fact, this issue of "non gauge theory states" in the prepotential Hilbert space $\mathcal{H}_p$ is related to the well known multiplicity problem in the direct products of SU(3). Actually, the states in (5.12) are not the states in the gauge theory Hilbert space, but no doubt are states in the Prepotential Hilbert space. Unlike SU(2) case (2.17), the SU(3) gauge theory Hilbert space $\mathcal{H}_g$ is contained in $\mathcal{H}_p$:

$$\mathcal{H}_g \subset \mathcal{H}_p. \tag{5.20}$$

Therefore, we now need projection operators to go from $\mathcal{H}_p$ to $\mathcal{H}_g$ such that the states in $\mathcal{H}_g$ are free from any spurious gauge invariant degrees of freedom. Hence the gauge theory Hilbert space, constructed out of the prepotentials must be free from this multiplicity problem. This makes SU(3) prepotential analysis slightly more involved than SU(2). In the next section we discuss the multiplicity problem for SU(3) reprresentation theory in detail.

5.3.1 The multiplicity problem in SU(3)

Note that the basis (5.12) in $\mathcal{H}_p$ is obtained by taking two direct products. The states $\hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L$ and $\hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R$ are individually direct products of quark and anti-quark irreducible representations: $(n_L = p, 0)_L \otimes (0, m_L = q)_L$ and $(n_R = q, 0)_R \otimes (0, m_R = p)_R$ respectively. Therefore, they can be further reduced using the SU(3) Clebsch Gordan series into irreps. of $SU_L(3)$ and $SU_R(3)$ respectively:

$$\begin{aligned}
(n_L = p, 0)_L \otimes (0, m_L = q)_L &= \sum_{\rho(L)=0}^{min(p,q)} \oplus \underbrace{(p-\rho(L), q-\rho(L))_L}_{\mathcal{H}^L_p(p-\rho(L),q-\rho(L),\rho(L))}, \\
(n_R = q, 0)_R \otimes (0, m_R = p)_R &= \sum_{\rho(R)=0}^{min(p,q)} \oplus \underbrace{(q-\rho(R), p-\rho(R))_R}_{\mathcal{H}^R_p(q-\rho(R),p-\rho(R),\rho(R))},
\end{aligned} \tag{5.21}$$

where, $\mathcal{H}^s_p(p-\rho(s), q-\rho(s), \rho(s))$, for $s = L/R$, denotes the Hilbert space of states (Young tableaues) constructed out of total p numbers of triplets (single boxes) and q numbers of anti-triplets (double boxes), out of which ρ numbers have combined to form singlet (column of double boxes). Hence, the states in $\mathcal{H}^s_p(p-\rho(s), q-\rho(s), \rho(s))$ transform as $(p-\rho(s), q-\rho(s))$ irrep of SU(3). An explicit example of (5.21) on a side $s = L/R$ of a link with $p = 1$ and $q = 1$ is as follows:

$$
\begin{aligned}
a^\dagger_\alpha(s)|0\rangle \otimes b^{\dagger\beta}(s)|0\rangle \equiv a^\dagger_\alpha(s)b^{\dagger\beta}(s)|0\rangle \;&=\; \underbrace{\left(a^\dagger_\alpha(s)b^{\dagger\beta}(s) - \frac{1}{3}\delta^\beta_\alpha(a\dagger(s)\cdot b^\dagger(s))\right)|0\rangle}_{(1-0,1-0)\in\mathcal{H}^s_p(1,1,0)} \\
&\quad + \underbrace{\frac{1}{3}\delta^\beta_\alpha(a^\dagger(s)\cdot b^\dagger(s))|0\rangle}_{(1-1,1-1)\in\mathcal{H}^s_p(0,0,1)} \qquad (5.22)
\end{aligned}
$$

Under SU(3) gauge transformations, the vectors in $\mathcal{H}^s_p(p,q,\rho)$ transform as $(p,q)_s$ irreducible representation of $SU_s(3)$ independent of the value of ρ $(= 0, 1, \cdots, \infty)$ leading to infinite multiplicity for each state. While the gauge theory Hilbert space $\mathcal{H}_g$ must always have $\rho = 0$ as the SU(3) invariant operators can never be created on a link by the action of Wilson loops as long as the vacuum is in $\mathcal{H}^s_p(p,q,\rho=0)$. Hence the gauge theory Hilbert space must be free from all sorts of multiplicity and should be exactly identified with $\mathcal{H}^s_p(p,q,\rho=0)$. We discuss the gauge theory hilbert space in detail in section 5.5.

The multiplicities associated with direct product representations of SU(3) have been extensively studied and classified in [40]. Following [40], the prepotential Hilbert space present at each side of a link can be subdivided into a set of irreducible Hilbert spaces each carrying all the irreps of SU(3). That is,

$$
\mathcal{H}^s_p(p,q) = \sum_{\rho=0}^{\min(p,q)} \mathcal{H}^s_p(p-\rho(s), q-\rho(s), \rho(s)), \quad s = L, R
$$

Hence, the complete prepotential Hilbert space for lattice gauge theory can be classified as:

$$
\begin{aligned}
\mathcal{H}_p = \prod_{\otimes link}\Big\{\mathcal{H}_p\Big\}_{link} \;&=\; \prod_{\otimes link}\Big\{\sum_{\rho=0}^{\infty}\sum_{p,q=0}^{\infty}\Big(\mathcal{H}^L_p(p,q,\rho)\otimes\mathcal{H}^R_p(q,p,\rho)\Big)\Big\}_{link} \\
&\equiv\; \prod_{\otimes link}\Big\{\sum_{\rho=0}^{\infty}\mathcal{H}_p(\rho)\Big\}_{link}. \qquad (5.23)
\end{aligned}
$$

5.4 The SU(3) invariant $Sp_L(2,R) \times Sp_R(2,R)$ algebra

Following [40], the multiplicity problem of SU(3) can be analyzed systematically with the introduction of the SU(3) invariant Sp(2,R) algebra. This analysis will enable us to solve the multiplicity problem and to identify the gauge theory Hilbert space at each end of a link on the lattice.

In order to identify the gauge theory Hilbert space $\mathcal{H}_g$ in (5.23), we define the following three color neutral operators on each side $s = L/R$ of every link:

$$k_-(s) \equiv a(s) \cdot b(s), \;\; k_+(s) \equiv a^\dagger(s) \cdot b^\dagger(s),$$
$$k_0(s) \equiv \frac{1}{2}\left(a^\dagger(s) \cdot a(s) + b^\dagger(s) \cdot b(s) + 3\right), \qquad s \equiv L, R. \tag{5.24}$$

These SU(3) color neutral operators satisfy the Sp(2,R) algebra on both sides of the link:

$$[k_0(s), k_\pm(s')] = \pm\delta_{s,s'} k_\pm(s), \qquad [k_-(s), k_+(s')] = 2\delta_{s,s'} k_0(s), s, s' \equiv L, R. \tag{5.25}$$

Further, as these Sp(2,R) generators are invariant under SU(3) transformations, they commute with the color electric fields. In other words:

$$[Sp_L(2,R) \otimes Sp_R(2,R), SU_L(3) \otimes SU_R(3)] = 0. \tag{5.26}$$

Therefore, the Hilbert space of SU(3) lattice gauge theory can be completely and uniquely labeled by $SU_L(3) \otimes Sp_L(2,R) \otimes SU_R(3) \otimes Sp_R(2,R)$ quantum numbers on every link. Consequently the SU(3) irreps constructed out of prepotentials at each end of the link must also be the irreps of Sp(2,R) located at that end of the link.

At the $s(\equiv L/R)$ end of the link the positive discrete irreps of $Sp_s(2,R)$ (only which are of our interest) are characterized by the normalized state:

$$|k,m\rangle_s \qquad \text{with, } k = 1/2, 1, 3/2, ... \;\&\;\; m = k, k+1, k+2, ...$$

where,

$$\left[\frac{1}{2}\left(k_+k_- + k_-k_+\right) - k_0^2\right]|k,m\rangle \;=\; k(1-k)|k,m\rangle \tag{5.27}$$
$$k_0(s)|k,m\rangle_s \;=\; m|k,m\rangle_s \tag{5.28}$$
$$k_\pm(s)|k,m\rangle_s \;=\; \sqrt{(m\pm k)(m\mp k\pm 1)}|k,m\pm 1\rangle_s \tag{5.29}$$

It follows from (5.29), after a few steps of algebra that:

$$|k,m\rangle_s = \sqrt{\frac{(2k-1)!}{(m-k)!(m+k-1)!}}\,(k_+(s))^{m-k}\,|k,k\rangle_s \tag{5.30}$$

$$\& \quad k_-(s)|k,k\rangle_s = 0. \tag{5.31}$$

As the generators of Sp(2,R) and SU(3) are mutually commuting, they must have a common set of eigenstates. Let us explicitly consider the state constructed in (5.16) which is an SU(3) irrep. One can explicitly check that the state $|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ in (5.16) satisfy the relation:

$$k_-|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = 0 \tag{5.32}$$

Now, this state must also be an irrep of Sp(2,R). To characterize the same state with Sp(2,R) quantum numbers, we calculate,

$$k_0|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \frac{1}{2}(p+q+3)|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \tag{5.33}$$

$$\begin{aligned}\left[\frac{1}{2}\left(k_+k_- + k_-k_+\right) - k_0^2\right]|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} &= \left[\frac{1}{2}[k_-,k_+] - k_0^2\right]|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \\ &= \left[\frac{1}{2}(p+q+3) - \frac{1}{4}(p+q+3)^2\right]|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \\ &= \frac{1}{2}(p+q+3)\left(1-\frac{1}{2}(p+q+3)\right)|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\end{aligned} \tag{5.34}$$

using the definition in (5.24). Now comparing (5.33) with (5.28) and (5.34) with (5.27), we identify:

$$|k,k\rangle \equiv |\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}. \tag{5.35}$$

Hence the states $|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ lie in the Hilbert space $\mathcal{H}_p(p,q,0)$. Similarly we also find that the states

$$(a^\dagger\cdot b^\dagger)^\rho|\psi\rangle^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \equiv |k,m\rangle$$

lying in $H_p(p,q,\rho)$ with $k=\frac{1}{2}(p+q+3)$ and $m=k+\rho$.

Coming back to the states in the prepotential Hilbert space $\mathcal{H}^s_p(p-\rho,q-\rho,\rho)$ on the s end of the link, they are as well the irreps of $Sp_s(2,R)$ and hence the eigenvalue of $k_0(s)$ on any state of this Hilbert space is $1/2(p+q+3)$. Hence, acting on the Hilbert space

$\mathcal{H}^s_p(p,q,\rho)$ in (5.23), $k_0(s)$ must have the eigenvalue $1/2(p+\rho+q+\rho+3) = 1/2(p+q+3)+\rho$. Thus the states in the sub spaces of the full prepotential Hilbert space defined in (5.23) are with the following Sp(2,R) quantum numbers:

$$m = \frac{1}{2}(p+q+3)+\rho, \quad \rho = 0,1,...\infty \tag{5.36}$$

Since the allowed values of m for a given k in an Sp(2,R) irrep are $k, k+1, k+2, ...,$ we can identify the SU(3) irreps in $\mathcal{H}^s_p(p,q,\rho)$ with $k = \frac{1}{2}(p+q+3)$. Hence the states $|k,m\rangle_s$ can be equivalently characterized as $|k,\rho\rangle_s \equiv |p,q,\rho\rangle_s$ which are again irreps of $SU_s(3)$.

Since the sum of the numbers of triplet and anti triplets are same at both the ends of a link (5.21), we get [40]: $k = k(L) = k(R) = \frac{1}{2}(p+q+3)$. Further $\rho(L) = \rho(R) = \rho$ appearing in (5.21) are the "magnetic quantum numbers" of $Sp_L(2,R) \otimes Sp_R(2,R)$. The raising (lowering) $k_+(s)(k_-(s))$ operators increase (decrease) the Sp(2,R) magnetic fluxes [40]:

$$|\mathcal{H}^L_p(p,q,\rho\pm 1)\rangle_L = k_\pm(s)\ |\mathcal{H}^L_p(p,q,\rho)\rangle,\ |\mathcal{H}^R_p(q,p,\rho\pm 1)\rangle_R = k_\pm(R)\ |\mathcal{H}^R_p(q,p,\rho)\rangle_R \tag{5.37}$$

Again one can see that using the λ matrix identity: $\sum_{a=1}^{8}\left(\frac{\lambda^a}{2}\right)^\alpha_\beta\left(\frac{\lambda^a}{2}\right)^\gamma_\sigma = \frac{1}{2}\delta^\alpha_\sigma\delta^\gamma_\beta - \frac{1}{6}\delta^\alpha_\beta\delta^\gamma_\sigma$, the squares of left and right electric fields can be written as:

$$\begin{aligned}
\sum_{a=1}^{8} E^a_L(n,i)E^a_L(n,i) &= \hat{N}(L)\left(\frac{\hat{N}(L)}{3}+1\right) + \hat{M}(L)\left(\frac{\hat{M}(L)}{3}+1\right) \\
&\quad - k_+(L)k_-(L) + \frac{1}{3}\hat{N}(L)\hat{M}(L) \\
\sum_{a=1}^{8} E^a_R(n,i)E^a_R(n,i) &= \hat{N}(R)\left(\frac{\hat{N}(R)}{3}+1\right) + \hat{M}(R)\left(\frac{\hat{M}(R)}{3}+1\right) \\
&\quad -k_+(R)k_-(R) + \frac{1}{3}\hat{N}(R)\hat{M}(R).
\end{aligned}$$

The electric field constraints (2.6) along with the $U(1)\otimes U(1)$ Gauss law constraints (5.10) imply:

$$k_+(L)k_-(L) = k_+(R)k_-(R). \tag{5.38}$$

On the other hand, the action of k_+k_- on a general Sp(2,R) irrep. $|k,m\rangle$ is given by [40]:

$$k_+k_-|k,m\rangle = (m-k)(m+k-1)|k,m\rangle, \tag{5.39}$$

where, $m = k+\rho$. In the present case the electric field constraint (5.38) and the eigenvalue equation (5.39) imply:

$$(m(L) - k(L))\,(m(L) + k(L) - 1) = (m(R) - k(R))\,(m(R) + k(R) - 1)\,. \qquad (5.40)$$

As $k(L) = k(R) = \frac{1}{2}(p+q+3)$, we get the unique solution of (5.40):

$$\rho_L(l) = \rho_R(l). \qquad (5.41)$$

Therefore, in the prepotential Hilbert space $\mathcal{H}_p$ the left and the right Sp(2,R) "magnetic" quantum numbers are same on every link.

Note that, in particular, the $\rho = 0$ Hilbert space without any "Sp(2,R) magnetic" flux in (5.21) is annihilated by k_-:

$$k_-(L)\ |\mathcal{H}_p^L(p,q,\rho=0)\rangle = 0, \qquad k_-(R)\ |\mathcal{H}_p^R(q,p,\rho=0)\rangle = 0. \qquad (5.42)$$

The equations (5.37) show that the "spurious gauge invariant" states in (5.18) are the vectors of one dimensional mutually orthogonal SU(3) invariant Hilbert spaces $\mathcal{H}_p^L(0,0,\rho) \otimes \mathcal{H}_p^R(0,0,\rho)$ with $\rho = 1, \cdots \infty$. The strong coupling vacuum is the $\rho = 0$ vacuum.

5.5 The SU(3) gauge theory Hilbert space $\mathcal{H}_g$

The various flux states in gauge theory Hilbert space $\mathcal{H}_g$ are created by the link matrices $U^\alpha{}_\beta$ acting on the strong coupling vacuum as in (2.10). Therefore, in order to identify $\mathcal{H}_g$ within $\mathcal{H}_p$ with Sp(2,R) structure (5.23), we now analyze the Sp(2,R) properties of the link operators in this section. We note that the link matrix $U^\alpha{}_\beta$ can not change the Sp(2,R) magnetic quantum number ρ. As shown at the bottom of Figure 5.2, $k_+(L) = a^\dagger(L)\cdot b^\dagger(L)$ and $k_+(R) = a^\dagger(R)\cdot b^\dagger(R)$ correspond to three Young tableau boxes in a vertical column (SU(3) singlets) on the left and right side of the links respectively. On the other hand, in terms of the link operators, this left and right anti-symmetrization on a link corresponds to: $\frac{1}{3!}\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon^{\beta_1\beta_2\beta_3}U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}U^{\alpha_3}{}_{\beta_3} = \det U \equiv 1$ or $tr\ (UU^\dagger) = 3$. Therefore, the states in $\mathcal{H}_g$, obtained by applying link operators on the strong coupling vacuum with $\rho = 0$ ($k_-(l)|0\rangle_l = 0,\ l = L, R$) will also carry $\rho = 0$ quantum numbers. In other words, they too will be annihilated by $k_-(l)$:

$$k_-(L)\Big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_r}{}_{\beta_r}\Big)|0\rangle = 0, \qquad k_-(R)\Big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_r}{}_{\beta_r}\Big)|0\rangle = 0. \quad (5.43)$$

Therefore, going back to the classification of $\mathcal{H}_p$ in (5.23), we identify:

$$\mathcal{H}_g \equiv \prod_{\otimes link} \left\{\mathcal{H}_p(\rho = 0)\right\}_{link} \equiv \mathcal{H}_p^0. \tag{5.44}$$

like in the case of SU(2) lattice gauge theory. In (5.44) $\mathcal{H}_g$ denotes $\rho = 0$ subspace of $\mathcal{H}_p$ satisfying $k_-|\psi\rangle = 0$ for $|\psi\rangle \in \mathcal{H}_g$. An example of $|\psi\rangle \in \mathcal{H}_g$ is the state given in (5.16) which reduces to the state $\left(a^\dagger_\alpha(s) b^{\dagger\beta}(s) - \frac{1}{3}\delta^\beta_\alpha (a^\dagger(s)\cdot b^\dagger(s))\right)|0\rangle$ with $s = L/R$ for $p = 1, q = 1$. Thus the kernel of $(k_-(L)k_-(R))$ in $\mathcal{H}_p$ is the SU(3) gauge theory Hilbert space $\mathcal{H}_g$. Further, (5.43) implies:

$$[k_-(L), U^\alpha{}_\beta] \simeq 0, \qquad\qquad [k_-(R), U^\alpha{}_\beta] \simeq 0, \tag{5.45}$$

In other words, $k_-(L)$ and $k_-(R)$ weakly commute with the link operators of SU(3) lattice gauge theory[2]. The symbol $\simeq$ in (5.45) implies that the commutators are zero only when they are applied on the vectors belonging to the gauge theory Hilbert space $\mathcal{H}_g \equiv \mathcal{H}_p^0$. We would now like to write the link operators in terms of SU(3) prepotential operators which create SU(3) fluxes only in the gauge theory Hilbert space $\mathcal{H}_g$. This is done in the next section.

5.6 SU(3) irreducible prepotential operators: Construction

In this section, we construct the SU(3) irreducible prepotential operators from the prepotential operators in (5.2) such that they directly create SU(3) irreducible fluxes exactly like in SU(2) case (2.16). This construction with all the it's group theoretical details is obtained in [34]. We define the SU(3) irreducible prepotential operators from prepotential operators such that:

1. they have exactly the same $SU(3) \otimes U(1) \otimes U(1)$ quantum numbers,
2. they commute with the Sp(2,R) destruction operator k_-.

As a result, acting on the strong coupling vacuum they directly create the gauge theory Hilbert space $\mathcal{H}_g$ completely bypassing the problem of spurious states like (5.18) in $\mathcal{H}_p$.

[2]Note that all the electric fields strongly commute with the Sp(2,R) generators (5.26).

we define SU(3) irreducible prepotentials [34] as:

$$\begin{aligned} A^\dagger_\alpha(L) &= a^\dagger_\alpha(L) + F_L\ k_+(L) b_\alpha(L), \qquad A^\dagger_\alpha(R) = a^\dagger_\alpha(R) + F_R\ k_+(R) b_\alpha(R), \\ B^{\dagger\alpha}(L) &= b^{\dagger\alpha}(L) + F_L k_+(L) a^\alpha(L), \qquad B^{\dagger\alpha}(R) = b^{\dagger\alpha}(R) + F_L k_+(R) a^\alpha(R). \end{aligned} \quad (5.46)$$

In (5.46), $A^\dagger_\alpha(L/R)$ are the triplets and $B^{\dagger\alpha}(L/R)$ are the anti-triplets. The factors F_L and F_R are given by:

$$F_L = -\frac{1}{N(L) + M(L) + 1}, \quad F_R = -\frac{1}{N(R) + M(R) + 1}. \quad (5.47)$$

These factors are chosen so that [34]:

$$\left[k_-(s), A^\dagger_\alpha(s)\right] \simeq 0; \qquad \left[k_-(s), B^{\dagger\alpha}(s)\right] \simeq 0. \quad (5.48)$$

for $s = L, R$. Taking dagger of the operators defined in (5.46) yields,

$$A^\alpha(s) \simeq a^\alpha(s) \ \ \& \ \ B_\alpha(s) \simeq b_\alpha(s) \quad (5.49)$$

for $s = L, R$, acting on $\mathcal{H}_g$. To get (5.49) from (5.46), we have used $k_- \simeq 0$. From (5.49), $k_-(s) \equiv A(s) \cdot B(s) \simeq a(s) \cdot b(s)$, implies that the annihilation operators in (5.49) strongly commute with $k_-(s)$.

This restriction (of weakly commuting) in (5.48) is made in order to obtain the traceless states (as in 5.16) only constructed by the action of these new operators on $\mathcal{H}_g$. In other words, for a state $|\psi\rangle \in \mathcal{H}_g$,

$$k_-|\psi\rangle \in \mathcal{H}_g = 0 \ \ \Rightarrow k_-\left(A^\dagger_\alpha|\psi\rangle \in \mathcal{H}_g\rangle\right) = 0 \ \& \ k_-\left(B^{\dagger\alpha}|\psi\rangle \in \mathcal{H}_g\rangle\right) = 0 \quad (5.50)$$

Using (5.46) and (5.47), it is easy to check that the irreducible Schwinger boson creation operators commute amongst themselves [3]

$$\left[A^\dagger_\alpha(s), A^\dagger_\beta(s')\right] = 0, \qquad \left[B^{\dagger\alpha}(s), B^{\dagger\beta}(s')\right] = 0, \qquad \left[A^\dagger_\alpha(s), B^{\dagger\beta}(s')\right] = 0. \quad (5.52)$$

[3]The other commutation relations acting on the SU(3) irreps. are [34]:

$$\begin{aligned} \left[A^\alpha(s), A^\dagger_\beta(s')\right] &\simeq \delta_{ss'}\left(\delta^\beta_\alpha - \frac{1}{N(s)+M(s)+2} B^{\dagger\alpha}(s) B_\beta(s)\right) \\ \left[A^\alpha(s), B^{\dagger\beta}(s')\right] &\simeq -\delta_{ss'}\frac{1}{N(s)+M(s)+2} B^{\dagger\alpha}(s) A^\beta(s) \\ \left[B_\alpha(s), B^{\dagger\beta}(s')\right] &\simeq \delta_{ss'}\left(\delta^\alpha_\beta - \frac{1}{N(s)+M(s)+2} A^\dagger_\alpha(s) A^\beta(s)\right) \end{aligned} \quad (5.51)$$

By construction, $A^\dagger_\alpha(s)$ and $B^{\dagger\alpha}(s)$ transform exactly like $a^\dagger_\alpha(s)$ and $b^{\dagger\alpha}(s)$, $s = L, R$ under $SU(3) \otimes U(1) \otimes U(1)$ and retain the same quantum numbers. Therefore, we can now define:

$$\left|{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}\right\rangle^0_L \otimes \left|{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}\right\rangle^0_R \equiv \hat{\mathcal{L}}\,{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \otimes \hat{\mathcal{R}}\,{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \tag{5.53}$$

where $|0\rangle_{L/R}$ are the strong coupling vacuum located at left and right end of a link as defined earlier. In (5.53), the additional Sp(2,R) quantum numbers $\rho_L = \rho_R = 0$ are put as superscript 0. The operators $\mathcal{L}$ and $\mathcal{R}$ are defined by replacing SU(3) prepotentials in L and R in (5.12) by the corresponding SU(3) irreducible prepotentials in (5.46), i.e.,

$$\hat{\mathcal{L}}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv A^\dagger_{\alpha_1}(L)\cdots A^\dagger_{\alpha_p}(L)B^{\dagger\beta_1}(L)\cdots B^{\dagger\beta_q}(L)|0\rangle_L,$$

and

$$\hat{\mathcal{R}}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \equiv A^\dagger_{\gamma_1}(R)\cdots A^\dagger_{\gamma_q}(R)B^{\dagger\delta_1}(R)\cdots B^{\dagger\delta_p}(R)|0\rangle_R.$$

In other words, the operators $\mathcal{L}$ and $\mathcal{R}$ in (5.53) are SU(3) irreducible unlike the L and R operators in (5.12) which are reducible according to (5.21). It can be shown that $\mathcal{L}$ and $\mathcal{R}$ are related to L and R by projection operators. In fact, these SU(3) flux creation operators $\mathcal{L}$ and $\mathcal{R}$ are the SU(3) analogues of the SU(2) flux creation operators $\mathcal{L}$ and $\mathcal{R}$ in (2.16) as both create irreducible fluxes. Further, like in SU(2) case, they bypass the problem of symmetrization and anti symmetrization associated with the link operators. This is because $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ in (5.53) are defined in terms of SU(3) irreducible prepotential operators which have all the symmetries of SU(3) Young tableauex inbuilt [34]. In other words the role played by SU(2) prepotentials in SU(2) lattice gauge theory is exactly equivalent to the role played by SU(3) irreducible prepotentials in SU(3) lattice gauge theory.[4]

Note that in terms of SU(3) irreducible prepotentials, the "spurious gauge invariant states" like in (5.18) or (5.19) do not exist as:

$$A^\dagger(L)\cdot B^\dagger(L)|0\rangle_L \equiv 0, \qquad\qquad A^\dagger(R)\cdot B^\dagger(R)|0\rangle_R \equiv 0. \tag{5.54}$$

[4]Note that, these Irreducible Schwinger bosons constructed as irreducible prepotentials are themselves quite rich in structure in the representation theoretic context. We exploit these objects to construct SU(3) coherent states [42] and to calculate SU(3) Clebsch Gordon coefficients [43] which are essential computational tool for loop formulation of gauge theories. We give both these construction in detail in the appendices A and B.

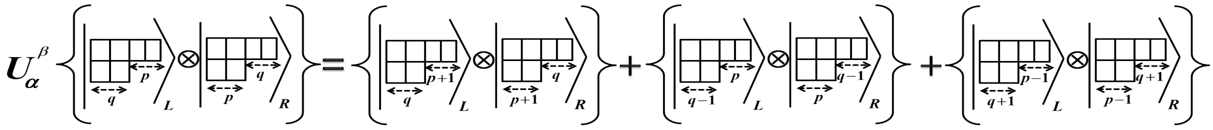

Figure 5.3: The Young tableau interpretation of the SU(3) link operator U in terms of the prepotential operators (5.57) acting on a state with p single boxes and q double boxes at the left of the link, and the conjugate state at the right. The three terms in (5.57) or (5.58) correspond to the three sets of (mutually conjugate) Young tableauex on the right hand side of this figure respectively. This is SU(3) generalization of Figure 2.2 for SU(2).

as these are nothing but the trace of the octet state or (1,1) state,

$$\left(a^\dagger_\alpha(s)b^{\dagger\beta}(s) - \frac{1}{3}\delta^\beta_\alpha(a\dagger(s)\cdot b^\dagger(s))\right)|0\rangle \tag{5.55}$$

defined at $s = L, R$. The other possible invariant constructed out of Irreducible prepotential also vanish as,

$$A(L)\cdot B(L)|0\rangle_L \simeq a(L)\cdot b(L)|0\rangle_L \equiv 0, \quad A(R)\cdot B(R)|0\rangle_L \simeq a(R)\cdot b(R)|0\rangle_L \equiv 0 \tag{5.56}$$

5.7 SU(3) link operators, electric fields and irreducible prepotentials

The SU(3) link operator must create 3 and 3^* fluxes at the left and right end of the link and should satisfy the $U(1)\otimes U(1)$ Gauss law constraints (5.10). These requirements are similar to SU(2) case discussed in chapter 2. In addition to this the link operator must satisfy the Sp(2,R) constraint given in (5.45). Anyway noting that the Sp(2,R) constraints are already solved in terms of irreducible Schwinger bosons in the last section and observing that, by construction, $A^\dagger_\alpha(s)$ and $B^{\dagger\alpha}(s)$ transform exactly like $a^\dagger_\alpha(s)$ and $b^{\dagger\alpha}(s)$, $l = s, R$, on a particular link the general structure of the link operator is:

$$U^\alpha_\beta = B^{\dagger\alpha}(L)\eta A^\dagger_\beta(R) + A^\alpha(L)\theta B_\beta(R) + \Big(B(L)\wedge A^\dagger(L)\Big)^\alpha\Big(A(R)\wedge B^\dagger(R)\Big)_\beta. \tag{5.57}$$

In (5.57), η, θ and δ are the SU(3) invariants and therefore can only depend on the number operators. These will be fixed later in this section. The link operator constructed in (5.57) has all the required group theoretical properties:

- Under SU(3) transformations $U(n,i)^{\alpha}{}_{\beta} \rightarrow (\Lambda_L)^{\alpha}{}_{\gamma} U(n,i)^{\gamma}{}_{\delta} (\Lambda_R{}^{\dagger})^{\delta}{}_{\beta}$.
- It is invariant under $U(1) \otimes U(1)$ abelian gauge transformations.
- It creates and destroys fluxes in $\mathcal{H}_g$ in (5.44). It is easy to check that the link operator $U^{\alpha}{}_{\beta}$ in (5.57) satisfy (5.45).
- Acting on a link state in $|p,q\rangle_L$ and $|q,p\rangle_R$ representations of $SU(3)_L \times SU(3)_R$:

$$\begin{aligned} U^{\alpha}{}_{\beta}|p,q\rangle_L \otimes |q,p\rangle_R &= C_1{}^{\alpha}{}_{\beta}|p+1,q\rangle_L \otimes |q,p+1\rangle_R + C_2{}^{\alpha}{}_{\beta}|p,q-1\rangle_L \otimes |q-1,p\rangle_R \\ &+ C_3{}^{\alpha}{}_{\beta}|p-1,q+1\rangle_L \otimes |q+1,p-1\rangle_R, \end{aligned} \tag{5.58}$$

  where C_1, C_2 and C_3 are the SU(3) Clebsch Gordan coefficients. The three terms in (5.57) correspond to the three terms in (5.58) respectively. In Figure 5.3, we illustrate (5.57) and (5.58) in terms of SU(3) Young tableau diagrams where the link operator changes the shape of the Young tableaux in all possible ways.

Written in matrix form the link operator has the form of a most general SU(3) matrix. Like in SU(2) case (2.21), it is convenient to break the full link operator matrix into left and right matrices as:

$$\begin{aligned} U &= \underbrace{\begin{pmatrix} B^{\dagger 1}(L)\eta_L & A^1(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^1\delta_L \\ B^{\dagger 2}(L)\eta_L & A^2(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^2\delta_L \\ B^{\dagger 3}(L)\eta_L & A^3(L)\theta_L & (B(L)\wedge A^{\dagger}(L))^3\delta_L \end{pmatrix}}_{U_L} \\ &\times \underbrace{\begin{pmatrix} A^1(R)\bar{\eta}_R & B^{1\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^1\bar{\delta}_R \\ A^2(R)\bar{\eta}_R & B^{2\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^2\bar{\delta}_R \\ A^3(R)\bar{\eta}_R & B^{3\dagger}(R)\bar{\theta}_R & (B(R)\wedge A^{\dagger}(R))^3\bar{\delta}_R \end{pmatrix}^{\dagger}}_{U_R} \equiv U_L U_R \end{aligned} \tag{5.59}$$

In (5.59), $\eta_L, \theta_L, \delta_L$ and $\bar{\eta}_R, \bar{\theta}_R, \bar{\delta}_R$ are the left and right invariants constructed out of the number operators. From (5.57):

$$\eta = \eta_L\ \eta_R, \qquad \theta = \theta_L\ \theta_R, \qquad \delta = \delta_L\ \delta_R. \tag{5.60}$$

Under SU(3) gauge transformation on a link,

$$\begin{aligned} & U \rightarrow \Lambda_L U \Lambda_R^{\dagger} = \Lambda_L U_L U_R \Lambda_R^{\dagger} \\ \Rightarrow\ & U_L \rightarrow \Lambda_L U \ \ \&\ \ U_R \rightarrow U_R \Lambda_R^{\dagger} \end{aligned} \tag{5.61}$$

From (5.59):

$$U_L^\dagger U_L =$$

$$= \begin{pmatrix} \bar\eta_L\Big(B\cdot B^\dagger\Big)\eta_L & \bar\eta_L\underbrace{\Big(B\cdot A\Big)}_{\simeq 0}\theta_L & \bar\eta_L\underbrace{\Big(B\cdot\big(B\wedge A^\dagger\big)\Big)}_{\equiv 0}\delta_L \\ \bar\theta_L\underbrace{\Big(A^\dagger\cdot B^\dagger\Big)}_{\simeq 0}\eta_L & \bar\theta_L\Big(A^\dagger\cdot A\Big)\theta_L & \bar\theta_L\underbrace{\Big(A^\dagger\cdot\big(B\wedge A^\dagger\big)\Big)}_{\equiv 0}\delta_L \\ \bar\delta_L\underbrace{\Big(\big(B^\dagger\wedge A\big)\cdot B^\dagger\Big)}_{\equiv 0}\eta_L & \bar\delta_L\underbrace{\Big(\big(B^\dagger\wedge A\big)\cdot A\Big)}_{\equiv 0}\theta_L & \bar\delta_L\Big(\big(A\wedge B^\dagger\big)\cdot\big(B\wedge A^\dagger\big)\Big)\delta_L \end{pmatrix} \quad (5.62)$$

Similarly,

$$U_R U_R^\dagger =$$

$$\begin{pmatrix} \eta_R\Big(A^\dagger\cdot A\Big)\bar\eta_R & \eta_R\underbrace{\Big(A^\dagger\cdot B^\dagger\Big)}_{\simeq 0}\bar\theta_R & \eta_R\underbrace{\Big(A^\dagger\cdot\big(B\wedge A^\dagger\big)\Big)}_{\equiv 0}\bar\delta_R \\ \theta_R\underbrace{\Big(B\cdot A\Big)}_{\simeq 0}\bar\eta_R & \theta_R\Big(B\cdot B^\dagger\Big)\bar\theta_R & \theta_R\underbrace{\Big(B\cdot\big(B\wedge A^\dagger\big)\Big)}_{\equiv 0}\bar\delta_R \\ \delta_R\underbrace{\Big(\big(B^\dagger\wedge A\big)\cdot A\Big)}_{\equiv 0}\bar\eta_R & \delta_R\underbrace{\Big(\big(B^\dagger\wedge A\big)\cdot B^\dagger\Big)}_{\equiv 0}\bar\theta_R & \delta_R\Big(\big(A\wedge B^\dagger\big)\cdot\big(B\wedge A^\dagger\big)\Big)\bar\delta_R \end{pmatrix} \quad (5.63)$$

In (5.62) and (5.63), we have suppressed the L/R indices from the prepotential operators $(A, A^\dagger)$ and $(B, B^\dagger)$. Note that, in (5.62) and (5.63), the off-diagonal elements vanishes. The weakly vanishing terms vanish using (5.54) and (5.56), whereas the strongly vanishing terms vanish as they contain anti-symmetric product of two identical operators. Demanding $U_L^\dagger U_L = 1$ and $U_R U_R^\dagger = 1$, we get:

$$\begin{aligned} \eta_L &= \frac{1}{\sqrt{B(L)\cdot B^\dagger(L)}}, \quad \theta_L = \frac{1}{\sqrt{A^\dagger(L)\cdot A(L)}}, \\ \delta_L &= \frac{1}{\sqrt{(A(L)\wedge B^\dagger(L))\cdot(B(L)\wedge A^\dagger(L))}}; \\ \eta_R &= \frac{1}{\sqrt{A^\dagger(R)\cdot A(R)}}, \quad \theta_R = \frac{1}{\sqrt{B(R)\cdot B^\dagger(R)}}, \\ \delta_R &= \frac{1}{\sqrt{(A(R)\wedge B^\dagger(R))\cdot(B(R)\wedge A^\dagger(R))}}. \end{aligned} \quad (5.64)$$

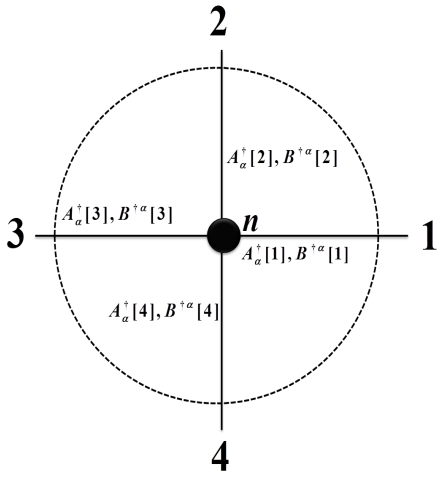

Figure 5.4: The SU(3) prepotentials associated with a lattice site n in $d = 2$. This is SU(3) generalization of Figure (3.4) for SU(2).

The link operators in (5.57) with (5.60) and (5.64) satisfy: $UU^\dagger = U^\dagger U = 1$. Having written the link operators in terms of the SU(3) irreducible prepotentials, we now cast the left and right electric fields (5.2) in terms of $A(l), A^\dagger(l), B(l), B^\dagger(l)$ with $l = L, R$. Using the very special structures of the SU(3) irreducible prepotentials in (5.46), explicit calculation shows that:

$$\begin{aligned} E^{\rm a}_L &= \left(a^\dagger(L)\frac{\lambda^{\rm a}}{2}a(L) - b(L)\frac{\lambda^{\rm a}}{2}b^\dagger(L)\right) \simeq \left(A^\dagger(L)\frac{\lambda^{\rm a}}{2}A(L) - B(L)\frac{\lambda^{\rm a}}{2}B^\dagger(L)\right) \\ E^{\rm a}_R &= \left(a^\dagger(R)\frac{\lambda^{\rm a}}{2}a(R) - b(R)\frac{\lambda^{\rm a}}{2}b^\dagger(R)\right) \simeq \left(A^\dagger(R)\frac{\lambda^{\rm a}}{2}A(R) - B(R)\frac{\lambda^{\rm a}}{2}B^\dagger(R)\right) \end{aligned} \quad (5.65)$$

In (5.65), the use of the expression of irreducible prepotentials given in (5.46) leads to explicit cancellation of the terms proportional to $b_\alpha \frac{\lambda^{\rm a}}{2}{}^\alpha{}_\beta a^{\dagger\beta}$. In fact the results (5.65) were expected because $(a^\dagger_\alpha, b^{\dagger\beta})$ and $(A^\dagger_\alpha, B^{\dagger\beta})$ have exactly the same $SU(3) \otimes U(1) \otimes U(1)$ transformation properties.

5.8 SU(3) gauge invariant states and Mandelstam Constraints in terms of Prepotentials

In this section we construct all possible SU(3) gauge invariant states at a given lattice site using prepotential approach. We also discuss the Mandelstam constraints which relate these gauge invariant states. As shown in Figure 5.4, every lattice site in 2d space dimension is associated with $2d$ pairs of quark-anti quark prepotentials $(A^{\dagger}_{\alpha}, B^{\dagger\alpha})$. Under a gauge transformation at site n, all these $2d$ quark (anti quark) prepotentials transform together as triplet (anti-triplet). Therefore, the fundamental SU(3) gauge invariant creation operator vertices at a lattice site n are:

$$\begin{aligned} L_{[ij]} &\equiv A^{\dagger}[i] \cdot B^{\dagger}[j], \qquad\qquad i \neq j, & (5.66)\\ A_{[i_1,i_2,i_3]} &= \epsilon^{\alpha_1\alpha_2\alpha_3} A^{\dagger}_{\alpha_1}[i_1] A^{\dagger}_{\alpha_2}[i_2] A^{\dagger}_{\alpha_3}[i_3], & (5.67)\\ B_{[j_1,j_2,j_3]} &= \epsilon_{\beta_1\beta_2\beta_3} B^{\dagger\beta_1}[j_1] B^{\dagger\beta_2}[j_2] B^{\dagger\beta_3}[j_3] \qquad i,j = 1,2,..,2d. & (5.68) \end{aligned}$$

These vertices are shown in Figure 5.5. We have taken $i \neq j$ in (5.66) because $L_{ii} = A^{\dagger}[i] \cdot B^{\dagger}[i] \simeq 0, \quad i,j = 1,2,\cdots 2d$ according to (5.54). Also, $A_{[i_1,i_2,i_3]}$ and $B_{[j_1,j_2,j_3]}$ are completely anti-symmetric in (i_1,i_2,i_3) and (j_1,j_2,j_3) indices respectively. The above

$$2d(2d-1) + 2({}^{2d}C_3) = \frac{2d(2d-1)(2d+1)}{3}$$

basic SU(3) gauge invariant operators enable us to write the most general SU(3) gauge invariant state at a given lattice site as:

$$|\vec{l}_{[ij]}, \vec{p}_{[i_1i_2i_3]}, \vec{q}_{[j_1j_2j_3]}\rangle = \prod_{\substack{i,j=1\\ i\neq j}}^{2d} \left(L_{[ij]}\right)^{l_{[ij]}} \prod_{[i_1i_2i_3]=1}^{{}^{2d}C_3} \left(A_{[i_1i_2i_3]}\right)^{p_{[i_1i_2i_3]}} \prod_{[j_1j_2j_3]=1}^{{}^{2d}C_3} \left(B_{[j_1j_2j_3]}\right)^{q_{[j_j i_2 j_3]}} |0\rangle. \quad (5.69)$$

In (5.69), $\vec{l}_{[ij]}, \vec{p}_{[i_1i_2i_3]}, \vec{q}_{[j_1j_2j_3]}$ are $\frac{2d(2d-1)(2d+1)}{3}$ non-negative integers describing all possible SU(3) gauge invariant states at a given lattice site. The various possible loop states set in pure SU(3) lattice gauge theory are direct products of (5.69) at various lattice sites consistent with $U(1)\otimes U(1)$ Gauss law (5.10) along every link. As in the loop formulation where various loop states are mutually related by Mandelstam constraints, not all states in (5.69) are linearly independent. In fact, in the present SU(3) prepotential formulation (like in SU(2) case) the Mandelstam constraints become local and take very simple forms

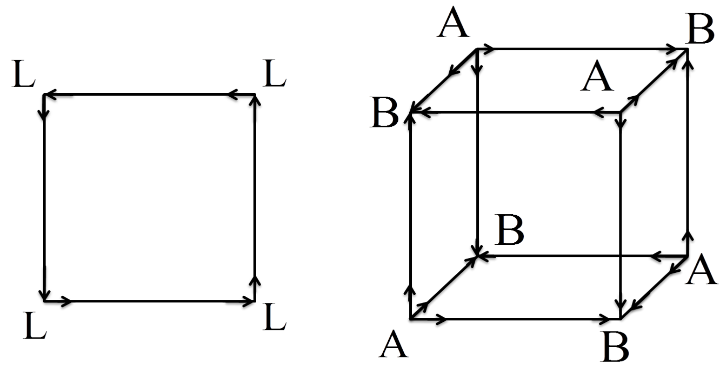

Figure 5.5: Graphical representation of the three possible SU(3) gauge invariant L, A, B types of vertices. Two simple $SU(3) \otimes U(1) \otimes U(1)$ gauge invariant loop states are also shown. The arrows represent the directions of the abelian (non-abelian) fluxes on the links (sites).

in terms of the SU(3) gauge invariant vertices in (5.66,5.67) and (5.68) at every lattice site n. We start with the simplest SU(3) Mandelstam constraints:

$$A_{[i_1,i_2,i_3]}B_{[j_1,j_2,j_3]} \equiv \sum_{\{s_1,s_2,s_3\}\in S_3} (-1)^s L_{[i_1 j_{s_1}]} L_{[i_2 j_{s_2}]} L_{[i_3 j_{s_3}]}. \tag{5.70}$$

In (5.70), S_3 denotes the permutation group of order 3, $\{s_1, s_2, s_3\}$ denote the 3! permutations of $\{1, 2, 3\}$ and s is the parity of permutation. In other words, the Mandelstam constraints (5.70) state that the A and B type vertices annihilate each other in pairs to produce L type vertices. The constraints (5.70) are illustrated in Figure 5.6. Therefore, the SU(3) gauge invariant states of $(L - A - B)$ type in (5.69) can always be written either as $(L - A)$ type or as $(L - B)$ type at each lattice site. Note that the L, A and B type invariants are the only invariants present at a site for SU(3) lattice gauge theory. Also they are related by a unique relation (5.70) which is analogue of the fundamental Mandelstam constraint for SU(2) obtained in terms of prepotentials as in (3.18). *The constraint (5.70) is basically the only fundamental Mandelstam constraint present locally at each site of the SU(3) lattice gauge theory which involves three loops passing through that site.* This fundamental Mandelstam identities were never discussed in the context of SU(3) lattice gauge theory. The set of Mandelstam identities well-known in the literature for SU(3) lattice gauge theory so far are due to Migdal [37]. These are basically the Man-

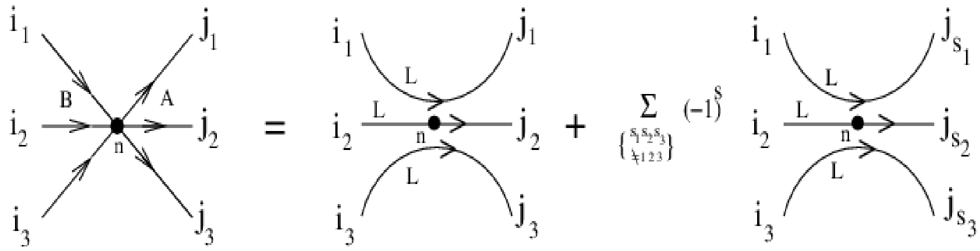

Figure 5.6: The graphical representation of local SU(3) Mandelstam constraints (5.70) in terms of SU(3) gauge invariant vertices A, B and L constructed out of the SU(3) irreducible prepotential operators at a lattice site n. The A and B type of vertices at n annihilate each other to produce L type of vertices.

delstam constraints (similar to the one given (3.8) for SU(2) case) in terms of the link operators, which are non-local, and involve more than three loops (as opposed to the one given in (5.70) involving only three loops). To illustrate this let us consider Migdal's set of constraints once more in the context of SU(3) lattice gauge theory. We consider the set of $r(>3)$ loops $\Gamma_1(n), \Gamma_2(n), \cdots, \Gamma_r(n)$, which passes through a particular site (n) from the direction $i_1, i_2, \ldots, i_r$ to the directions $j_1, j_2, \ldots, j_r$ respectively, then they satisfy the identity:

$$\sum_{\substack{\alpha_{i_1}\cdots\alpha_{i_r}\\ \beta_{j_1}\cdots\beta_{j_r}}} \epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}}\epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} \left(W(\Gamma_1(n))^{\alpha_{j_1}}{}_{\beta_{i_1}}\right) \left(W(\Gamma_2(n))^{\alpha_{j_2}}{}_{\beta_{i_2}}\right) \cdots \left(W(\Gamma_r(n))^{\alpha_{j_r}}{}_{\beta_{i_r}}\right) \equiv 0. \quad (5.71)$$

Note that, in terms of the original link variables, the loops $\Gamma_1(n), \Gamma_2(n), \cdots, \Gamma_r(n)$ are allowed to be as large as one wishes and r can also be as large as possible. *However, in terms of the prepotentials, the constraints (5.71) become local.* All one has to do is to replace the Wilson loops $W(\Gamma_s)^{\alpha_{j_s}}{}_{\beta_{i_s}}$ in (5.71) by the prepotentials which are attached to the links in the direction i_s and j_s corresponding to the starting and ending direction at that particular point, i.e,:

$$W(\Gamma_s)^{\alpha_{j_s}}{}_{\beta_{i_s}} \rightarrow L^{\alpha}{}_{\beta}[i_s.j_s] \equiv B[j_s]^{\dagger\alpha} A^{\dagger}[i_s]_{\beta}, \quad s = 1, 2, \cdots, r. \quad (5.72)$$

Note that unlike non-local Wilson loop $W(\Gamma_s)$, the operators $B^{\alpha}(j_s)$ and $A^{\dagger}_{\beta}(i_s)$ and hence $L^{\alpha}{}_{\beta}[i_s.j_s]$ are completely defined locally at lattice site n. Noting that $Tr\left(L^{\alpha}{}_{\beta}[i_s.j_s]\right) = L_{[i_s j_s]}$, the non-local Mandelstam constraints (5.71) acquire the following simple local

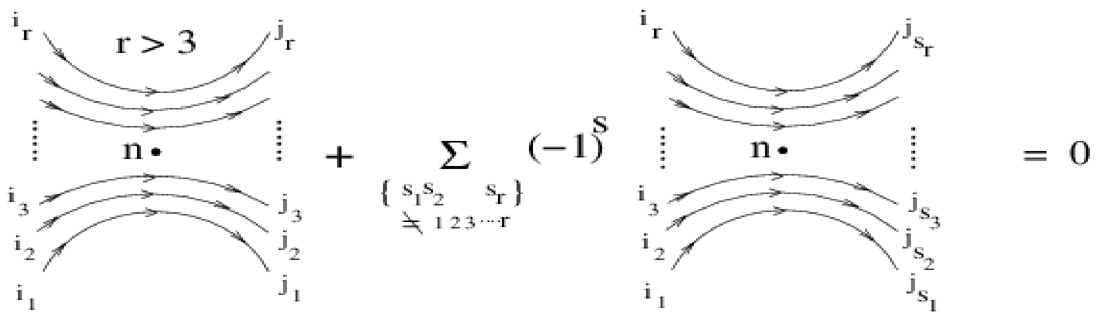

Figure 5.7: The graphical representation of local SU(3) Mandelstam constraints (5.73) involving only L type of vertices at lattice site n.

form:

$$\sum_{\{s_1,s_2,\cdots s_r\}\in S_r} (-1)^s L_{[i_{s_1}j_{s_1}]} L_{[i_{s_2}j_{s_2}]} \cdots L_{[i_{s_r}j_{s_r}]} = 0 \tag{5.73}$$

and are illustrated in Figure 5.7. *Note that all the unnecessary details like shapes, sizes and lengths of the loops* $\Gamma_1, \Gamma_2, \cdots \Gamma_r$ *in (5.71) disappear in the corresponding prepotential form (5.73).*

The Mandelstam constraints in their present local prepotential forms (5.70) and (5.73), instead of non-local form (3.8) in terms of link operators, are now accessible to explicit local solutions like in SU(2) lattice gauge theory [29]. Note that they are still infinite in number at every lattice site but the number of Mandelstam constraints are finite. The solution of these fundamental Mandelstam constraint will be the orthonormal physical Hilbert space constructed out of prepotentials. The solutions must be all possible mutually independent linear combinations of the states in (5.69) at a given lattice site. Following the techniques discussed in [44] in the context of duality transformations in lattice gauge theories, these linear combinations can be obtained by characterizing the resultant states at a site n by their complete SU(3) quantum numbers with the net SU(3) fluxes being zero. This will be SU(3) analogue of SU(2) results. The quantum numbers needed to specify such states can be easily computed [44] as follows. In the standard language [45], the SU(3) irreducible representations are completely specified by 5 quantum numbers: $|p, q, i, i_z, y\rangle$ where p and q are the eigenvalues of two SU(3) Casimir operators and i, i_z, y are the SU(3) "magnetic" quantum numbers representing isospin, it's third component and hyper charge respectively. In the present language with constraints, each of the 2d directions (see Figure 5.4 for d=2) is associated with 6 harmonic oscillators $(A^\dagger[i], B^\dagger[i], i = 1, 2, \cdots, 2d)$ and therefore requires 6 occupation numbers to completely specify the basis. The constraints $k_-[i] \equiv a[i] \cdot b[i] \simeq 0$ reduces this to 5 in each direction. Therefore $5 \times 2d = 10d$ quantum numbers are needed to specify a local Hilbert space basis completely at each lattice site. Not all these quantum numbers are independent as 2d of these are related to the previous sites by $U(1) \otimes U(1)$ Gauss law constraints (5.10). Therefore, we are left with $8d$ quantum numbers at every lattice site. Finally, the SU(3) gauge invariance further implies 8 constraints. Therefore, the net independent quantum numbers are $8(d-1)$ per lattice site. As expected, this is the the number of transverse degree of freedom of 8 SU(3) gluons in d spatial dimensions at every lattice site. The abelian $U(1) \times U(1)$ fluxes over the links will now glue these local SU(3) invariant orthogonal basis at neighboring lattice sites according to their Gauss laws (5.10). This will give complete solutions of all the SU(3) Mandelstam constraints like what was done in SU(2) lattice gauge theory [29].

5.9 Summary and discussion

In this chapter we analyze SU(3) lattice gauge theory in terms of the prepotential operators which under gauge transformations transform like fundamental matter fields. We constructed the SU(3) irreducible prepotential operators which acting on strong coupling vacuum directly created the QCD fluxes around lattice sites. All SU(3) gauge invariant vertices in terms of these QCD flux operators were constructed at every lattice site. These SU(3) invariant vertices, in turn, enabled us to cast all SU(3) Mandelstam constraints in their local forms. As discussed in the text, this is an essential step towards their complete solution. The complete solution of Mandelstam constraints, in turn, will allow us to write down SU(3) lattice gauge theory completely and exactly in terms of minimum essential gauge invariant loop and string co-ordinates without any redundant loop/strings degrees of freedom as in the SU(2) case. The prepotential operators also allow us to simplify lattice gauge theory Hamiltonian (4.1). In particular, for the present SU(3) case, one can simply replace the plaquette or magnetic term $TrU_{plaquette}$ in (4.1) by a new plaquette interaction consisting of the 4 L type vertices at the 4 corners of every plaquette. Note that the new Hamiltonian constructed this was has exactly the same symmetries as (4.1). The addition of matter field interactions in the prepotential formulation is trivial as matter and prepotential have similar SU(3) gauge transformation properties. The difference lies in the abelian $U(1) \otimes U(1)$ transformations under which matter fields remain invariant.

In the next chapter we again generalize the idea of irreducible prepotentials to arbitrary SU (N) gauge theory.

Chapter 6

Prepotential Formulation for SU(N) Lattice Gauge Theory

In the last two chapters we have developed prepotential formulation of lattice gauge theory for gauge group SU(2) and SU(3) respectively. At this stage it is natural to generalize this to arbitrary SU(N). The first obstacle comes from the group theoretic complications that are inevitable for SU(N) representations. We have earlier constructed SU(3) irreducible Schwinger bosons and utilized them to construct SU(3) irreducible prepotentials. Now, this SU(3) construction does not seem to have direct generalization to SU(N). This is because unlike SU(3), SU(N) representation (N>3) are constructed out of more than two fundamental irreps. We require (N-1) fundamental irreps to construct any arbitrary representation of SU(N) [45]. The increasing number of fundamental irreps for SU(N) (with increasing N) representations increases the multiplicity of SU(N) irreps. In this chapter we take account of all these multiplicity and define a new set of modified irreducible Schwinger bosons for SU(N) [35], $N \geq 2$. Note that all of the earlier SU(2) and SU(3) results can also be recovered from this new construction. These general SU(N) irreducible Schwinger boson operators defined at each site of lattice can be identified with SU(N) irreducible prepotential operators of SU(N) lattice gauge theory. We also show that in terms of these new irreducible prepotentials one can construct local gauge invariant operators at each lattice site. The non-local loop states are constructed by weaving the local SU(N) invariant states by

$$\underbrace{U(1) \otimes U(1) \otimes \ldots \otimes U(1)}_{N-1 \text{ terms}} \equiv U(1)^{(N-1)}$$

abelian gauge transformation. The Mandelstam constraints can also be realized in this new setting. It is interesting to see a whole class of Mandelstam constraints become trivial in these new formulation.

6.1 Prepotentials in SU(N) Lattice Gauge Theory

In this section we define the SU(N) prepotential operators which carries the fundamental representation of SU(N). The Schwinger boson representation of SU(N) group is obtained by $N-1$ independent harmonic oscillator N-plets each transforming as N representation of SU(N). In terms of these prepotentials residing at both the left and right end of a link (we do not put a explicit link index as long as we are on a same link), the left and right electric fields are respectively constructed as,

$$\begin{aligned} E_L &= a^\dagger(1,L)\frac{\Lambda^{\rm a}}{2}a(1,L) + a^\dagger(2,L)\frac{\Lambda^{\rm a}}{2}a(2,L) + \ldots + a^\dagger(N-1,L)\frac{\Lambda^{\rm a}}{2}a(N-1,L) \\ E_R &= a^\dagger(1,R)\frac{\Lambda^{\rm a}}{2}a(1,R) + a^\dagger(2,R)\frac{\Lambda^{\rm a}}{2}a(2,R) + \ldots + a^\dagger(N-1,R)\frac{\Lambda^{\rm a}}{2}a(N-1,R) \end{aligned} \quad (6.1)$$

In (6.1), $\Lambda^{\rm a}$ is the generalized Gell-Mann matrices for SU(N) [45], $a^\dagger[i,s]$ for $i = 1, 2, \ldots N-1$ and $s = L, R$ are harmonic oscillator N-plets satisfying the following harmonic oscillator algebra:

$$\left[a^\alpha[i,s], a^\dagger_\beta[j,s']\right] = \delta^\alpha{}_\beta \delta_{ij}\delta_{ss'} \quad (6.2)$$

The electric fields given in (6.1) generates SU(N) gauge thansformation at both the ends of a link. Hence there are $N-1$ Casimirs for rank $N-1$ group SU(N), present at both the ends of a link. In terms of prepotentials these Casimirs are expressed as the $N-1$ number operators. i.e

$$N[i,s] = a^\dagger[i,s]\cdot a[i,s] \quad (6.3)$$

for $i = 1, 2, \ldots N-1$ and $s = L, R$. The eigenvalues of the Casimir operatots are to be denoted by n^s_i. The SU(N) irreps are characterized by the eigen values of these $N-1$ number of Casimir operators. In terms of Young tableaux an arbitrary SU(N) irrep is represented as given in figure 6.1 and 6.2. The SU(N) Casimirs are the count of the number of boxes in each of the $N-1$ rows of the general SU(N) Young tableaux. The constituting fundamental irreps of the tableaux are columns of $1, 2, 3, \ldots, N-1$ boxes respectively. Hence to specify a particular irrep of SU(N), one must specify the number of each constituting fundamental irrep, which is basically the difference in length of a particular row and that of the next lower row as shown in figure 6.1 and 6.2. Concentrating on a link of the spatial lattice, the SU(N) irreducible states residing at each end of a link is characterized by

$$|n_1 - n_2, n_2 - n_3, \ldots, n_{N-2} - n_{N-1}, n_{N-1}\rangle_{L/R} \tag{6.4}$$

It is worth mentioning here that, to be a valid irrep of SU(N), the $N-1$ Casimir eigenvalues of SU(N) at s end of a link must satisfy the relation:

$$n_1^s \geq n_2^s \geq n_3^s \geq \ldots \geq n_{N-1}^s. \tag{6.5}$$

The above relation also carries the information that the i^{th} row of Young tableaux containing n_i^s boxes have been created by $a^\dagger[i, s]$ only as shown in figure 6.3. The prepotential operators defined in (6.1) transform as a N representation under SU(N),

$$a^\dagger_\alpha[i, s] \rightarrow a^\dagger_\beta[i, s] \left(\Lambda^\dagger_s\right)^\beta{}_\alpha \tag{6.6}$$

Note that, unlike the SU(3) analysis in last section, here all the prepotentials do transform in the same way under SU(N) as N-plets and are represented by single boxes of the Young tableaux. But we need $N-1$ different fundamental representations of SU(N), to construct all possible irreps (like $4, 6, 4^*$ for SU(4)). However all other fundamental representations of SU(N) are constructed out of these fundamental N-plets. Hence all $N-1$ fundamental representations of SU(N) are obtained as antisymmetric combination of k, for $k = 1, 2, \ldots N-1$ independent N-plets.

6.2 The Additional Abelian Gauge Invariance

Like SU(2) and SU(3) case, from the defining relation of prepotentials (6.1) it is evident that the generators of gauge transformations are invariant under the following abelian gauge transformation:

$$a^\dagger[i, s] \rightarrow e^{i\theta[i,s]} a^\dagger[i, s] \tag{6.7}$$

for $s = L, R$. As there are $N-1$ Schwinger bsosons at each end of the link, we have the following additional symmetry corresponding to each side of a link:

$$\underbrace{U(1) \otimes U(1) \otimes \ldots \otimes U(1)}_{N-1 \text{ terms}} \equiv U(1)^{(N-1)}. \tag{6.8}$$

Hence, for a particular link there is an extra $U(1)^{2(N-1)}$ invariance which was not present in the old Kogut-Susskind formulation of Hamiltonian lattice gauge theory. However, this extra abelian symmetry group is actually of a smaller rank as all these two sets of $U(1)^{(N-1)}$ transformations are not independent. This is because of the following facts:

1. The gauge theory Hilbert space spanned by the states in (6.4) is created by the action of link operators, i.e

$$U_{\alpha_n}{}^{\beta_n} \ldots U_{\alpha_2}{}^{\beta_2} U_{\alpha_1}{}^{\beta_1} |0\rangle \tag{6.9}$$

From the transformation properties of link variables it is manifest that whichever representation is created at the left of a link the conjugate representation of the same must be created at right end.

2. The left and right electric fields are not independent but they are related by parallel tranport through link operator U satisfying the constraint:

$$E_L^2 = E_R^2 \tag{6.10}$$

The first one of the two above facts can be understood well by considering explicitly the corresponding Young tableaux. Let's consider the left Hilbert space in spanned by the states:

$$|n_1 - n_2, n_2 - n_3, \ldots, n_{N-2} - n_{N-1}, n_{N-1}\rangle_L$$

represented by the Young tableaux as in figure 6.1: According to 1, it's conjugate representation will be created at the other end of the link and is represented by the Young tableaux in figure 6.2. The conjugate representation in terms of Young tableaux means that combining (i.e joining from opposite directions) these two Young tableaux one would get a tableaux consisting only of columns of length N. The condition of two Young tableaux to be conjugate of each other is illustrated in figure 6.3.

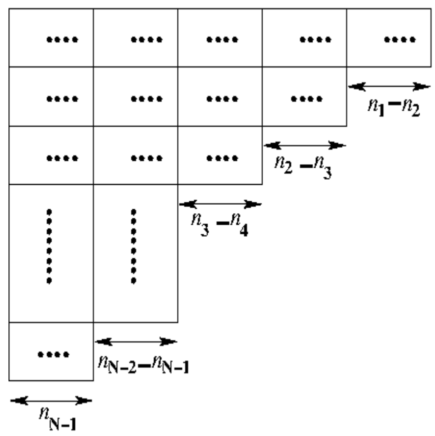

Figure 6.1: General SU(N) Young tableaux at the left end of a link

As stated earlier, the $N-1$ SU(N) Casimir operators at each end of a link basically counts the number of boxes in each row of a Young tableaux. We also follow the convention that one particular type harmonic oscillators ($a^\dagger[i,s]$) creates Young tableaux boxes in a particular row (i^{th} row). With reference to fig 6.1 and 6.2, the two Young tableau created by the action of U's at both the ends of a link, which are conjugate to each other must satisfy the following relations between their $2(N-1)$ Casimir eigenvalues:

$$\begin{aligned}
&n_1^L = n_1^R = n \\
&n_2^L + n_{N-1}^R = n \\
&n_3^L + n_{N-2}^R = n \\
&\vdots \\
&n_p^L + n_{N-p+1}^R = n \\
&\vdots \\
&n_{N-2}^L + n_3^R = n \\
&n_{N-1}^L + n_2^R = n
\end{aligned} \tag{6.11}$$

These constraints are also illustrated in figure 6.3. The above set of constraints are the SU(N) analogues of (2.19) and (5.10) for SU(2) and SU(3) cases respectively. Hence for SU(N) we have only $N-1$ independent Casimir eigenvalues for both the left and right

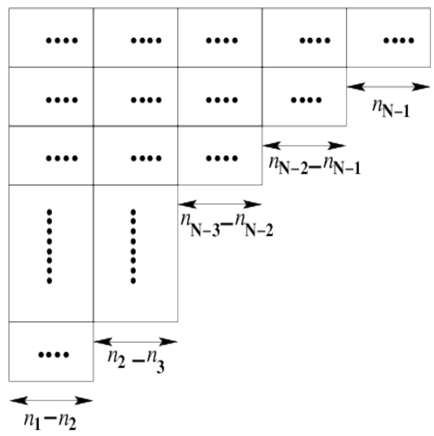

Figure 6.2: Conjugate SU(N) Young tableaux at the right end of the link

representations as below.

$$|n_1 - n_2, n_2 - n_3, \ldots, n_{N-2} - n_{N-1}, n_{N-1}\rangle_L \otimes |n_{N-1}, n_{N-2} - n_{N-1}, \ldots, n_2 - n_3, n_1 - n_2\rangle_R$$

Above is the state created by the action of link operators on a link and is characterized by $N-1$ Casimir eigenvalues of SU(N). At this point, we put $N = 3$ in the above set of constraints given in (6.11) and get,

$$n_2^L + n_2^R = n_1^L = n_1^R, \tag{6.12}$$

putting this in (6.12), we get the state,

$$|n_1 - n_2, n_2\rangle_L \otimes |n_2, n_1 - n_2\rangle_R$$

on a link, which is compatible with the constraint given in (5.10) for SU(3) in the earlier picture.

The set of constraints in (6.11), are nothing but the reflection of the Electric field constraint (6.10). Thus for prepotential formulation of SU(N) gauge theory we would have an additional set of $N-1$ number of U(1) Gauss law constraints as all the U(1) gauge transformation parameters given in (6.7) are not independent, but they are related

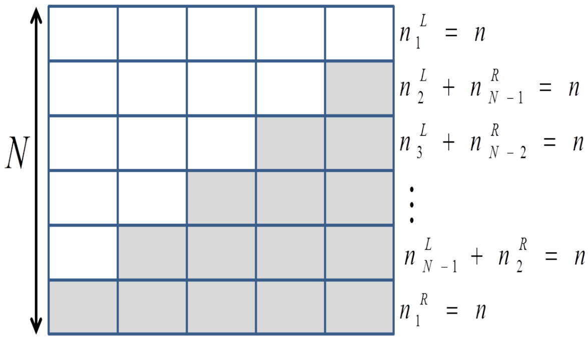

Figure 6.3: Conditions of left and right irreps for being mutually conjugate. The Young tableaux consisting of white boxes is the left SU(N) irrep and the Young tableaux constructed by inverting the set of grey boxes vertically is the right SU(N) irrep. The number of boxes in each of the irreps are constrained in order to make mutually conjugate irreps.

as,

$$\begin{aligned}
&\theta[1, L] = -\theta[1, R] \\
&\theta[2, L] + \theta[N-1, R] = 0 \\
&\theta[3, L] + \theta[N-2, R] = 0 \\
&\vdots \\
&\theta[p, L] + \theta[N-p+1, R] = 0 \\
&\vdots \\
&\theta[N-2, L] + \theta[3, R] = 0 \\
&\theta[N-1, L] + \theta[2, R] = 0
\end{aligned} \tag{6.13}$$

Hence the total $U(1)^{2(N-1)}$ invariance given in (6.7) is reduced to $U(1)^{(N-1)}$ invariance which is to be satisfied by the prepotentials.

6.3 SU(N) prepotential Hilbert space vs. SU(N) Gauge theory Hilbert space

As discussed in detail for SU(3) case, the prepotential Hilbert space $\mathcal{H}_P$ spanned by the basis vectors $|n_1^L, n_2^L, ..., n_{N-1}^L\rangle \otimes |n_1^R, n_2^R, ..., n_{N-1}^R\rangle$ is actually a much bigger space than the

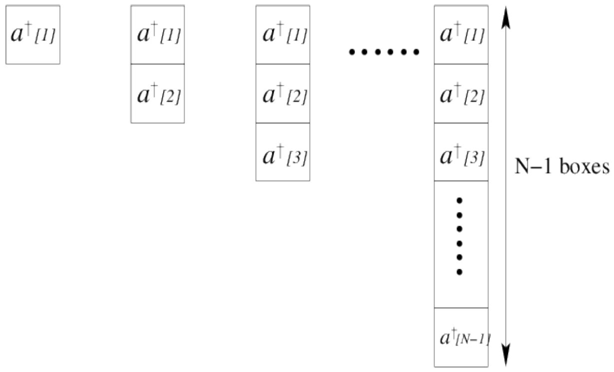

Figure 6.4: A general SU(N) Young tableaux consists of $N-1$ fundamental Young tableax columns. Each of the fundamental columns of length $l = 1, 2, .., N-1$ are again antisymmetric combination of r independent Young tableaux boxes.

Gauge theory Hilbert space created by the action of U's on the strong coupling vacuum. Note that prepotential basis vectors are direct product of $N-1$ number of different N representations of SU(N). Whereas, the link variables create mutually conjugate irreducible representations of SU(N) at both the ends of it. To create all possible irrep of SU(N) one needs all the $N-1$ fundamental representations of it which is to be constructed out of these $N-1$ prepotentials as shown in the Young tableaux in figure 6.3. As a property of Young tableaux it is always symmetric along a row and antisymmetric along a column. It is worth specifying again that we stick to the convention that the i^{th} row of an Young tableaux at side $s = L, R$ is always created by $a^\dagger[i,s]$. This convention automatically brings the constraint $n_i^s \geq n_j^s$ for $i < j$, where, n_i^s is the eigenvalue of $a^\dagger[i,s] \cdot a[i,s]$, to construct valid Young tableaux. Taking into account this number operator constraints the vertical antisymmetry of any Young tableaux can be implemented by imposing the following constraints [35, 46]:

$$\mathcal{C}_{ij}[L] \otimes \mathcal{C}_{kl}[R] \equiv \left(a^\dagger[i,L] \cdot a[j,L]\right) \otimes \left(a^\dagger[k,R] \cdot a[l,R]\right) \approx 0, \qquad \text{for } i < j \ \& \ k < l \tag{6.14}$$

These constraints are basically SU(N) invariants. Note that, all the invariants of SU(N) are as follows:

$$\left(a^\dagger[i,L] \cdot a[j,L]\right) \otimes \left(a^\dagger[k,R] \cdot a[l,R]\right) \quad \forall i,j,k,l \tag{6.15}$$

where, for $i = j$ and $k = l$ the invariants are trivial number operators. To illustrate the constraint (6.14) let us consider the special case of SU(3) irreps created at both the ends of a link for SU(3) gauge theory. For SU(3), we need two fundamental irreps 3 and 3^*. Instead we can also consider two different 3-representation and create 3^* as the antisymmetric combination of these two. In terms of prepotentials we must have $a^\dagger[i,s]$, for $i = 1,2$ and $s = L,R$. We also choose the convention that the 1st row of Young tableaux or the 3 representation on side s is created by $a^\dagger[1,s]$. The second row of Young tableaux is created by $a^\dagger[2,s]$. To construct an SU(3) irrep, whenever $a^\dagger[2,s]$ appears, it must be anti-symmetrized with one of the $a^\dagger[1,s]$. This implies that to construct any particular irrep of SU(3), number of $a^\dagger[2,s]$ must be less than or equal to that of $a^\dagger[1,s]$. Moreover, the vertical antisymmetry implies:

$$a^\dagger[1,s] \cdot a[2,s] \simeq 0 \tag{6.16}$$

The weak equation implies acting on any SU(3) irrep. Hence, for a particular link of SU(3), we have,

$$a^\dagger[1,L] \cdot a[2,L] \otimes a^\dagger[1,R] \cdot a[2,R] \simeq 0. \tag{6.17}$$

In case of SU(4) the above illustration also holds to create 4 and 6 representation. Together with these two a third fundamental irrep and correspondingly a third set of prepotential operators to create the 4^* representation. Hence, the additional constraints arising for SU(4) are:

$$\begin{aligned} a^\dagger[1,L] \cdot a[3,L] \otimes a^\dagger[1,R] \cdot a[3,R] \simeq 0 \\ a^\dagger[2,L] \cdot a[3,L] \otimes a^\dagger[2,R] \cdot a[3,R] \simeq 0. \end{aligned} \tag{6.18}$$

These are the analogous equations of (6.14) for $N = 3,4$.

Since the link operators acting on a state in the gauge theory Hilbert space creates a linear combination of states in the same Hilbert space, it must commute with these constraints also. i.e

$$\left[\mathcal{C}_{ij}[L] \otimes \mathcal{C}_{kl}[R], {U_\alpha}^\beta\right] \approx 0 \tag{6.19}$$

In the next section we construct the link operators which directly create the SU(N) fluxes at both the ends of a lattice and also satisfies all the previously mentioned constraints.

6.4 SU(N) link operator in terms of prepotentials

In this section we will construct SU(N) link operator in terms of prepotential operators. As discussed in the last chapter for the case of SU(3), the link operator acting on the strong coupling vacuum creates the states in the gauge theory Hilbert space. It was also discussed that the action of link operator on strong coupling vacuum cannot create any SU(N) invariant state on a particular link itself. Thus starting from its action on vacuum it can never produce any gauge invariant state on a link. The same argument is valid for arbitrary SU(N) link operators too. The SU(N) invariant operators in (6.15) for $i > j$ acting on strong coupling vacuum creates the states corresponding to the $(a^\dagger \cdot b^\dagger)|0\rangle$ on a link for SU(3) case discussed in last chapter. As in the SU(3) case these states are never created by the action of link operators. Hence to construct the link operator in terms of prepotential we must cut down these spurious gauge invariant degrees of freedom on a link as it was done in the last section for SU(3) case. Following the same strategy as SU(3) case, the first step of the construction of link operators in SU(N) lattice gauge theory in terms of prepotentials is to construct irreducible prepotential operators which is free from any spurious gauge invariant degrees of freedom on a link. Moreover these irreducible prepotentials should directly create SU(N) irreducible representation having all the symmetry properties of a general SU(N) Young tableaux. Like in SU(3) case, we exploit these irreducible prepotentials to construct the link operators which create the left and right irreps, mutually conjugate to each other, belonging to the gauge theory Hilbert space and finally satisfying all the $U(1)^{\otimes(N-1)}$ constraints.

6.4.1 Irreducible Prepotential Operators

Let us define irreducible prepotential operators residing at each lattice site, with the following properties [35]

- carries all the same quantum numbers as that of ordinary prepotential operators defined in (6.1),
- commutes with the constraints $\mathcal{C}_{ij}[s] \approx 0$,
- acts as a monomial on the vacuum to create irreducible representations.

Let us first discuss the construction of the irreducible prepotentials for SU(3) and SU(4), and then we will generalize it to arbitrary SU(N). Concentrating on a particular side of the link $s = L/R$, the prepotential operators present are $a^\dagger[i,s], a[i,s]$, $i = 1,2$ for SU(3) and $i = 1,2,3$ for SU(4) to construct any arbitrary irrep of SU(3) as well as SU(4). The constraints that any irrep must satisfy are:

$$a^\dagger[1,s]\cdot a[2,s] \simeq 0 \tag{6.20}$$

for SU(3), together with

$$\begin{aligned} a^\dagger[1,s]\cdot a[3,s] &\simeq 0 \\ a^\dagger[2,s]\cdot a[3,s] &\simeq 0 \end{aligned} \tag{6.21}$$

for SU(4). Clearly all the prepotential operators do not commute with these constraints. The ones which commute is $a^\dagger[1,s]$ and $a[3,s]$. Hence we define a set of irreducible prepotential operators all of which do commute with all the constraint equations as follows:

$$\begin{aligned} A^{\dagger\alpha}[1,s] &= a^{\dagger\alpha}[1,s] \\ A^{\dagger\alpha}[2,s] &= a^{\dagger\alpha}[2,s] + F_1^2[s]\left(a^\dagger[2,s]\cdot a[1,s]\right)a^{\dagger\alpha}[1,s] \\ A^{\dagger\alpha}[3,s] &= a^{\dagger\alpha}[3,s] + F_1^3[s]\left(a^\dagger[3,s]\cdot a[1,s]\right)a^{\dagger\alpha}[1,s] \\ &\quad + F_2^3[s]\left(a^\dagger[3,s]\cdot a[2,s]\right)a^{\dagger\alpha}[2,s] \\ &\quad + F_2^3[s]F_1^2[s]\left(a^\dagger[3,s]\cdot a[2,s]\right)\left(a^\dagger[2,s]\cdot a[1,s]\right)a^{\dagger\alpha}[1,s]. \end{aligned} \tag{6.22}$$

where,

$$F_i^k = -\frac{1}{N[i,s]-N[k,s]+1+k-i} \quad . \tag{6.23}$$

is calculated from the condition that,

$$[C_{ij}[s], A^\dagger[j,s]] \simeq 0, \qquad \forall j \tag{6.24}$$

where, $C_{ij}[s] = a^\dagger[i,s]\cdot a[j,s]$. Note that, in construction (6.22), the transformation properties of new operators have been kept exactly same as the original ones by keeping the total number of $a^\dagger[i,s]$ in $A^\dagger[i,s]$ equal to one whereas everything else equal to zero.

Similarly we also define the prepotential annihilation operators as:

$$
\begin{aligned}
A_\alpha[3,s] &= a_\alpha[3,s] \\
A_\alpha[2,s] &= a_\alpha[2,s] + H_2^3[s]\left(a^\dagger[3,s]\cdot a[2,s]\right)a_\alpha[3,s] \\
A_\alpha[1,s] &= a_\alpha[1,s] + H_1^3[s]\left(a^\dagger[3,s]\cdot a[1,s]\right)a_\alpha[3,s] \\
&\quad + H_1^2[s]\left(a^\dagger[2,s]\cdot a[1,s]\right)a_\alpha[2,s] \\
&\quad + H_2^3[s]H_1^2[s]\left(a^\dagger[3,s]\cdot a[2,s]\right)\left(a^\dagger[2,s]\cdot a[1,s]\right)a_\alpha[3,s].
\end{aligned}
\tag{6.25}
$$

where,

$$
H_k^i[s] = \frac{1}{N[i,s]-N[k,s]+1+k-i} \equiv -F_i^k[s] \quad . \tag{6.26}
$$

are calculated such that the annihilation operators too commute with the constraints. Now we generalize the above construction to arbitrary SU(N). Let us define [35] the left and right prepotentials $A^\dagger[i,s]$, for $i = 1,2,..,N-1$ and $s = L, R$ as:

$$
A^{\dagger\alpha}[k,s] = a^{\dagger\alpha}[k,s] + \sum_{r=1}^{k-1}\ \sum_{\{i_1,..,i_r\}=1}^{k-1\,\prime} F_{i_1}^k[s]\ F_{i_2}^k[s]\cdots F_{i_r}^k[s]\mathcal{C}^\dagger_{i_1k}\ \mathcal{C}^\dagger_{i_2i_1}\ldots\mathcal{C}^\dagger_{i_ri_{r-1}}a^{\dagger\alpha}[i_r,s] \tag{6.27}
$$

In (6.27) $k = 1,2,\cdots(N-1)$ and the prime over the second summation ($\sum'$) implies that the ordering $k > i_1 > i_2 > ... > i_r$ has to be maintained. Note that for $k = 2,3$ it reduces to (6.22) for SU(4) case. The general form of $F_i^k[s]$ is obtained in [35] as

$$
F_i^k = -\frac{1}{N[i,s]-N[k,s]+1+k-i} \quad . \tag{6.28}
$$

Similarly all the $N-1$ fundamental irreducible annihilation operators for SU(N) can also be constructed. The general k^{th} annihilation operator for SU(N) at side s is given by,

$$
A_\alpha[k,s] \equiv a_\alpha[k,s] + \sum_{r=1}^{N-1}\ \sum_{\{\{i_r\}=k+1\}}^{N-1\,\prime} H_k^{i_1}[s]H_k^{i_2}[s]..H_k^{i_r}[s]\mathcal{C}^\dagger_{ki_1}[s]\mathcal{C}^\dagger_{i_1i_2}[s]..\mathcal{C}^\dagger_{i_{r-1}i_r}[l]a_\alpha[i_r,s]. \tag{6.29}
$$

In (6.29) $k = 1,2,\cdots(N-1)$ and the prime over the second summation ($\sum'$) implies that the ordering $k < i_1 < i_2 < ... < i_r < N-1$ has to be maintained. The coefficients are obtained as:

$$
H_k^i[s] = \frac{1}{N[i,s]-N[k,s]+1+k-i} \equiv -F_i^k[s] \quad . \tag{6.30}
$$

Again note that, the construction (6.29) reduces to (6.25) for SU(4) case with $k = 1, 2, 3$. Explicit computation shows that all non-trivial SU(N) invariant operators (as given in (6.15), for $i \neq j$ & $k \neq l$ in terms of ordinary prepotential operators), constructed out of irreducible prepotentials becomes proportional to the constraints and hence acting on any SU(N) irrep vanishes. Hence the Hilbert space created by SU(N) irreducible prepotential operators contains all SU(N) representations and every representation appears once as:

$$A^{\dagger}[i,s] \cdot A[j,s] \approx 0, \quad \forall \ \ i \neq j \ \& \ s = L, R \ . \tag{6.31}$$

The only remaining SU(N) invariant operators in terms of SU(N) irreducible prepotentials are $A^{\dagger}[i,s] \cdot A[i,s], \ i = 1, 2, \cdots, (N-1)$. These operators, being weakly related to the SU(N) Casimir operators $N[i,l]$, do not lead to multiplicity. In terms of the irreducible prepotentials residing at both the ends of a link, the SU(N) irreps at that particular link is nothing but the following monomial operator acting on prepotential vacuum:

$$\begin{aligned}
&\left|\alpha_1^{[1]}, \alpha_2^{[1]}, \ldots, \alpha_{n_1}^{[1]}; \alpha_1^{[2]}, \alpha_2^{[2]}, \ldots, \alpha_{n_2}^{[2]}; \cdots, \alpha_1^{[N-1]}, \alpha_2^{[N-1]} \ldots, \alpha_{n_{N-1}}^{[N-1]}\right\rangle_s \qquad (6.32) \\
\equiv\ & \hat{\mathcal{S}}|0\rangle \\
=\ & \underbrace{\left(A^{\dagger\alpha_1^{[N-1]}}[N-1,s] \cdots A^{\dagger\alpha_{n_{N-1}}^{[N-1]}}[N-1,s]\right)}_{n_{N-1} \ of A^{\dagger}[N-1,s]} \cdots\cdots \\
& \underbrace{\left(A^{\dagger\alpha_1^{[2]}}[2,s] \cdots A^{\dagger\alpha_{n_2}^{[2]}}[2,s]\right)}_{n_2 \ of A^{\dagger}[2,s]} \underbrace{\left(A^{\dagger\alpha_1^{[1]}}[1,s] \cdots A^{\dagger\alpha_{n_1}^{[1]}}[1,s]\right)}_{n_1 \ of A^{\dagger}[1,s]} |0\rangle
\end{aligned}$$

where $s = L, R$ and $\hat{\mathcal{S}} = \hat{\mathcal{L}}/\hat{\mathcal{R}}$.

6.4.2 SU(N) link operators in terms of irreduicible prepotentials

The SU(N) link operator must transform as N at left and as N^* at right end of the link and should also satisfy $U(1)^{(N-1)}$ Gauss law given in (6.11) as well as the constraints in (6.19). The last one is already solved in terms of irreducible prepotentials. In this section we construct the link operator exploiting the irreducible prepotentials which satisfy the $U(1)^{(N-1)}$ constraints (6.11) and produce two mutually conjugate flux states at both of its ends. The general tensor structure of U compatible with $U(1)^{(N-1)}$ Gauss law in (6.11) is as follows:

$$\begin{aligned} U_\alpha{}^\beta &= A^\dagger_\alpha[1,L]\ \mathcal{N}_1\ \left(A^\dagger[N-1,R]\wedge A^\dagger[N-2,R]\wedge\ldots\wedge A^\dagger[2,R]\wedge A^\dagger[N-1,R]\right)^\beta \\ &+\ A^\dagger_\alpha[2,L]\ \mathcal{N}_2\ A^\beta[N-1,R] \\ &+\ A^\dagger_\alpha[3,L]\ \mathcal{N}_3\ A^\beta[N-2,R] \\ &\ \ \vdots \\ &+\ A^\dagger_\alpha[N-1,L]\ \mathcal{N}_{N-1}\ A^\beta[2,R] \\ &+\ \left(A[1,L]\wedge A[2,L]\wedge\ldots\wedge A[N-1,L]\right)_\alpha\ \mathcal{N}_N\ A^\beta[1,R] \end{aligned} \tag{6.33}$$

This link operator acting on the mutually conjugate states (represented by Young tableaux) at both the ends of the link deforms both the irreps in all possible way (so that it is still an irrep) but keeping the mutual conjugacy intact. The action of U on a general mutually conjugate state

$$|n_1,n_2,n_3,\ldots n_{N-1}\rangle_L\otimes|n_1,n_1-n_{N-1},n_1-n_{N-2},\ldots,n_1-n_2\rangle_R$$

is as follows:

$$\begin{aligned} &U^\beta_\alpha\left(|n_1,n_2,\ldots n_{N-1}\rangle_L\otimes|n_1,n_1-n_{N-1},\ldots,n_1-n_2\rangle_R\right) \\ &=\mathcal{N}_1\underbrace{|n_1+1,n_2,\ldots n_{N-1}\rangle_L\otimes|n_1+1,n_1-n_{N-1}+1,\ldots,n_1-n_2+1\rangle_R}_{n_1\to n_1+1} \\ &+\mathcal{N}_2\underbrace{|n_1,n_2+1,\ldots n_{N-1}\rangle_L\otimes|n_1,n_1-n_{N-1},\ldots,n_1-n_2-1\rangle_R}_{n_2\to n_2+1} \\ &+\mathcal{N}_3\underbrace{|n_1,n_2,n_3+1,\ldots n_{N-1}\rangle_L\otimes|n_1,n_1-n_{N-1},\ldots,n_1-n_3-1,n_1-n_2\rangle_R}_{n_3\to n_3+1} \\ &\qquad\vdots \\ &+\mathcal{N}_{N-1}\underbrace{|n_1,n_2,n_3,\ldots n_{N-1}+1\rangle_L\otimes|n_1,n_1-n_{N-1}-1,n_1-n_{N-2},\ldots,n_1-n_2\rangle_R}_{n_{N-1}\to n_{N-1}+1} \\ &+\mathcal{N}_N\underbrace{|n_1-1,n_2-1,\ldots n_{N-1}-1\rangle_L\otimes|n_1-1,n_1-n_{N-1},\ldots,n_1-n_2\rangle_R}_{n_i\to n_i-1\ \forall i} \end{aligned} \tag{6.34}$$

The coefficients $\mathcal{N}_i$ are now to be fixed from the unitarity of link operator, i.e from the condition $UU^\dagger = U^\dagger U = 1$. For calculational simplicity let us split the full link operator to its left and right parts. Infact like the SU(2) and SU(3) case, for arbitrary SU(N) also the full U matrix can be written as a product of a left matrix U_L and a right matrix U_R as follows;

$$U_\alpha{}^\beta = (U_L)_\alpha{}^\gamma (U_R)_\gamma{}^\beta \tag{6.35}$$

Where,

$$U_L = \begin{pmatrix} A_1^\dagger[1,L]\mathcal{N}_1^L & \dots & A_1^\dagger[N-1,L]\mathcal{N}_{N-1}^L & (A[1,L]\wedge..\wedge A[N-1,L])_1\mathcal{N}_N^L \\ A_2^\dagger[1,L]\mathcal{N}_1^L & \dots & A_2^\dagger[N-1,L]\mathcal{N}_{N-1}^L & (A[1,L]\wedge..\wedge A[N-1,L])_2\mathcal{N}_N^L \\ \vdots & \ddots & \vdots & \vdots \\ A_N^\dagger[1,L]\mathcal{N}_1^L & \dots & A_N^\dagger[N-1,L]\mathcal{N}_{N-1}^L & (A[1,L]\wedge..\wedge A[N-1,L])_N\mathcal{N}_N^L \end{pmatrix}$$

and

$$U_R^T = \begin{pmatrix} \mathcal{N}_1^R\left(A^\dagger[1,R]\wedge..\wedge A^\dagger[N-1,R]\right)^1 & \mathcal{N}_2^R A^1[N-1,R] & \dots & \mathcal{N}_N^R A^1[1,R] & \\ \mathcal{N}_1^R\left(A^\dagger[1,R]\wedge..\wedge A^\dagger[N-1,R]\right)^2 & \mathcal{N}_2^R A^2[N-1,R] & \dots & \mathcal{N}_N^R A^2[1,R] & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{N}_1^R\left(A^\dagger[1,R]\wedge..\wedge A^\dagger[N-1,R]\right)^N & \mathcal{N}_2^R A^N[N-1,R] & \dots & \mathcal{N}_N^R A^N[1,R] & \end{pmatrix}$$

where, from (6.33) $\mathcal{N}_i = \mathcal{N}_i^L\mathcal{N}_i^R$ for all $i = 1,2,\dots N$. These coefficients are to be calculated from the unitarity property. Let us first calculate,

$$\begin{aligned} U_L^\dagger U_L = \;& \text{diag}\Big((\mathcal{N}_1^L)^2 A[1,L]\cdot A^\dagger[1,L],\;\; (\mathcal{N}_2^L)^2 A[2,L]\cdot A^\dagger[2,L],\dots\dots \\ & ,(\mathcal{N}_{N-1}^L)^2 A[N-1,L]\cdot A^\dagger[N-1,L], \\ & (\mathcal{N}_N^L)^2 (A^\dagger[N-1,L]\wedge..\wedge A^\dagger[1,L])\cdot(A[1,L]\wedge..\wedge A[N-1,L])\Big) \end{aligned} \tag{6.36}$$

All the off-diagonal terms are zero due to one of the following reasons:

$$\begin{aligned} & A[i,L]\cdot A^\dagger[j,L]\approx 0 \quad ,i\neq j(\text{ from (6.31)}) \\ & \left(A^\dagger[N-1,L]\wedge..\wedge A^\dagger[1,L]\right)\cdot A^\dagger[i,L]\equiv 0, \qquad \forall i \\ & A[i,L]\cdot(A[1,L]\wedge..\wedge A[N-1,L])\equiv 0, \qquad \forall i \end{aligned} \tag{6.37}$$

To make all the diagonal entries of $U_L^\dagger U_L$ equal to 1, the coefficients are fixed as,

$$
\begin{aligned}
(\mathcal{N}_1^L) &= \frac{1}{\sqrt{A[1,L]\cdot A^\dagger[1,L]}} \\
(\mathcal{N}_2^L) &= \frac{1}{\sqrt{A[2,L]\cdot A^\dagger[2,L]}} \\
\vdots \quad & \quad\quad \vdots \\
(\mathcal{N}_{N-1}^L) &= \frac{1}{\sqrt{A[N-1,L]\cdot A^\dagger[N-1,L]}} \\
(\mathcal{N}_N^L) &= \frac{1}{\sqrt{(A^\dagger[N-1,L]\wedge..\wedge A^\dagger[1,L])\cdot(A[1,L]\wedge..\wedge A[N-1,L])}}
\end{aligned} \tag{6.38}
$$

Similar calculation with $U_R U_R^\dagger$ gives,

$$
\begin{aligned}
U_R U_R^\dagger &= \mathrm{diag}\Big((\mathcal{N}_1^R)^2(A^\dagger[N-1,R]\wedge..\wedge A^\dagger[1,R])\cdot(A[1,R]\wedge..\wedge A[N-1,R]), \\
&\quad (\mathcal{N}_2^R)^2 A[N-1,R]\cdot A^\dagger[N-1,R], \\
&\quad (\mathcal{N}_3^R)^2 A[N-2,R]\cdot A^\dagger[N-2,R],\ldots\ldots,(\mathcal{N}_N^R)^2 A[1,R]\cdot A^\dagger[1,R]\Big) \\
&\equiv \mathrm{diag}(1,1,1,\ldots\ldots,1) \\
\Rightarrow \quad & \mathcal{N}_1^R = \frac{1}{\sqrt{(A^\dagger[N-1,R]\wedge..\wedge A^\dagger[1,R])\cdot(A[1,R]\wedge..\wedge A[N-1,R])}} \\
& \mathcal{N}_2^R = \frac{1}{\sqrt{A[N-1,R]\cdot A^\dagger[N-1,R]}} \\
& \mathcal{N}_3^R = \frac{1}{\sqrt{A[N-2,R]\cdot A^\dagger[N-2,R]}} \\
& \vdots \quad\quad \vdots \\
& \mathcal{N}_N^R = \frac{1}{\sqrt{A[1,R]\cdot A^\dagger[1,R]}}
\end{aligned}
$$

We write the final ${U_\alpha}^\beta$ as follows:

$$
\begin{aligned}
&U_\alpha{}^\beta = \\
&A^\dagger_\alpha[1,L]\ \frac{1}{\sqrt{A[1,L]\cdot A^\dagger[1,L]}}\frac{1}{\sqrt{(A^\dagger[N-1,R]\wedge..\wedge A^\dagger[1,R])\cdot(A[1,R]\wedge..\wedge A[N-1,R])}} \\
&\quad \left(A^\dagger[N-1,R]\wedge A^\dagger[N-2,R]\wedge\ldots\wedge A^\dagger[2,R]\wedge A^\dagger[N-1,R]\right)^\beta \\
&+\ A^\dagger_\alpha[2,L]\ \frac{1}{\sqrt{A[2,L]\cdot A^\dagger[2,L]}}\frac{1}{\sqrt{A[N-1,R]\cdot A^\dagger[N-1,R]}}\ A^\beta[N-1,R] \\
&+\ A^\dagger_\alpha[3,L]\ \frac{1}{\sqrt{A[3,L]\cdot A^\dagger[3,L]}}\frac{1}{\sqrt{A[N-2,R]\cdot A^\dagger[N-2,R]}}\ A^\beta[N-2,R] \\
&\quad \vdots \\
&+\ A^\dagger_\alpha[N-1,L]\ \frac{1}{\sqrt{A[N-1,L]\cdot A^\dagger[N-1,L]}}\frac{1}{\sqrt{A[2,R]\cdot A^\dagger[2,R]}}\ A^\beta[2,R] \\
&+\ (A[1,L]\wedge A[2,L]\wedge\ldots\wedge A[N-1,L])_\alpha \\
&\frac{1}{\sqrt{(A^\dagger[N-1,L]\wedge..\wedge A^\dagger[1,L])\cdot(A[1,L]\wedge..\wedge A[N-1,L])}}\frac{1}{\sqrt{A[1,R]\cdot A^\dagger[1,R]}}\ A^\beta[1,R]
\end{aligned}
\tag{6.39}
$$

6.5 Loop States and Mandelstam Constraints in terms of Prepotentials

In terms of prepotential operators for gauge group SU(3) or general SU(N), we find the counting of local loop degrees of freedom exactly same way as done in the case of SU(2) and SU(3). As the rank of the SU(N) group is $N-1$, there will be $N-1$ number of different prepotential operators associated with each end of the link. Hence, for a site in d dimensional lattice, where $2d$ links meet, we will have total $2d(N-1)$ prepotentials. The SU(N) invariants are antisymmetric combination of N number of prepotentials. Hence there exist ${}^{2d(N-1)}C_N$ basic local gauge invariant operators constructed out of prepotentials around every lattice site. Moreover, in terms of prepotentials there exist an additional $U(1)^{(N-1)}$ abelian gauge invariance (6.11) in the theory for each direction, which will put $(N-1)d$ additional constraints at a site for SU(N) gauge theory. Hence we have total

$$\mathcal{M} = {}^{2d(N-1)}C_N - (N-1)d$$

independent gauge invariant quantum numbers to characterize states locally at a particular site. This implies that all these quantum numbers are not independent as there should be only $\mathcal{N} = (N^2 - 1)(d - 1)$ number of physical degrees of freedom per lattice site (see section 3.1). Thus there are $(\mathcal{M} - \mathcal{N})$ redundant loop degrees of freedom due to the overcompleteness of the Wilson loops at each site. These loop redundancy is reflected into the Mandelstam constraints. Hence there should be exactly $(\mathcal{M} - \mathcal{N})$ number of Mandelstam constraints among the local loop states.

To understand the invariants constructed at a site, we consider a site of a d dimensional lattice for SU(N) lattice gauge theory, each link carries $(N - 1)$ number of independent fundamental representations of SU(N), each of which is a single Young tableaux box. The invariants for SU(N) means a column of N Young tableaux boxes. Hence it is possible to construct a column of N Young tableaux boxes from available $2d(N - 1)$ boxes. Such a state is given by,

$$T_{[I_1 I_2 \ldots I_N]} = \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_N} A^{\dagger \alpha_1}[I_1] A^{\dagger \alpha_2}[I_2] \ldots A^{\dagger \alpha_N}[I_N] \tag{6.40}$$

where, I_n's are all different prepotentials present around a site. One can choose such a state in ${}^{2d(N-1)}C_N$ different way. Hence the most general gauge invariant state can be written as

$$|\vec{p}_{[I_1 I_2 \ldots I_3]}\rangle = \prod_{[I_1 I_2 \ldots I_N]=1}^{{}^{2d}C_N} \left(T_{[I_1 I_2 \ldots I_N]}\right)^{p_{[I_1 I_2 \ldots I_N]}} |0\rangle. \tag{6.41}$$

Where, $\vec{p}$ is vector of dimension ${}^{2d(N-1)}C_N$ with positive integer entry. Note that these kind of gauge invariant objects are analogous to the one defined in (5.67) and (5.68) for SU(3). The another type of gauge invariant operator defined in (5.66) for SU(3) do not exist here by construction. The most general loop state in (6.41) is SU(N) analogue of the loop state (3.28) for SU(2) and (5.69) for SU(3) respectively. However, all these basic gauge invariant variables $T_{[I_1 .. I_N]}$ are not independent. To illustrate this, let us consider the $N = 3$ case as an example.

For SU(3) lattice gauge theory, each end of a link carries two different fundamental triplet representations (in the earlier description in chapter 4 it carries a triplet and an anti-triplet). Hence, at a site in d dimensional lattice, there exist total $2d \times 2$ triplets or prepotential operators $A^{\dagger \alpha}[I]$, with $I = 1, 4d$ attached to 2d links. The only gauge invariant operators constructed out of these prepotentials at particular site are:

$$T_{[I_1 I_2 I_3]} = \epsilon_{\alpha \beta \gamma} A^{\dagger \alpha}[I_1] A^{\dagger \beta}[I_2] A^{\dagger \gamma}[I_3] \tag{6.42}$$

where, I_1, I_2, I_3 denotes any three different prepotential operators among the available $4d$ prepotentials present at that site. These invariants are similar to the operators constructed in (5.67), (5.68) for SU(3) case in the last chapter. Thus the number of possible gauge invariant operators in SU(3) lattice gauge theory at each site is ${}^{4d}C_3$. Note that, these invariants for SU(N) theory with $N \geq 3$ are exactly same as the invariants constructed for SU(2) gauge theory in chapter 3. We have observed that, SU(2) invariants are mutually related by the identity given in (3.18). In the same way, exploiting the identity,

$$\epsilon_{\alpha\beta\gamma}\epsilon_{lmn} = \delta_{\alpha l}\delta_{\beta m}\delta_{\gamma n} - \delta_{\alpha m}\delta_{\beta l}\delta_{\gamma n} + \delta_{\alpha m}\delta_{\beta n}\delta_{\gamma l} - \delta_{\alpha n}\delta_{\beta m}\delta_{\gamma l} + \delta_{\alpha n}\delta_{\beta l}\delta_{\gamma m} - \delta_{\alpha l}\delta_{\beta n}\delta_{\gamma m} \quad (6.43)$$

we find the following relation between the invariant operators (exactly equivalent to the SU(2) identity (3.18)):

$$\begin{aligned} & T_{[I_1I_2I_3]}T_{[I_4I_5I_6]} - T_{[I_1I_2I_4]}T_{[I_3I_5I_6]} + T_{[I_1I_2I_5]}T_{[I_3I_4I_6]} \\ & -T_{[I_1I_2I_6]}T_{[I_3I_4I_5]} + T_{[I_1I_3I_4]}T_{[I_2I_5I_6]} + T_{[I_1I_4I_5]}T_{[I_2I_3I_6]} \\ & -T_{[I_1I_3I_5]}T_{[I_2I_4I_6]} + T_{[I_1I_5I_6]}T_{[I_2I_3I_4]} - T_{[I_1I_4I_6]}T_{[I_2I_3I_5]} + T_{[I_1I_3I_6]}T_{[I_2I_4I_5]} = 0. \end{aligned} \quad (6.44)$$

where $I_1 \dots I_6$ denotes any six different prepotential triplets chosen out of available $2 \times 2d$ operators present around the site. Note that, *(6.44) is indeed the fundamental Mandelstam identity for SU(3) involving only 3 loops passing through a particular site.* This procedure can be generalized in a straightforward way to arbitrary SU(N) to obtain fundamental Mandelstam identities involving only N loops passing through a site.

The other non-local Mandelstam identities in terms of the link operators (the one given in (5.71)) still exists with SU(N) theory for $(r > N)$ loops passing through a particular site. Similar to the SU(3) case, here also, this nonlocal constraints can be completely analyzed locally using prepotential formalism (as done in (5.72)), i,e:

$$W \to T_{[I_1I_2\dots I_N]} \quad (6.45)$$

as $T_{[I_1I_2\dots I_N]}$'s are the only SU(N) invariants constructed out of the prepotentials which are as well local. But the identities in terms of Wilson loops are not fundamental as all of these for any aritrary number of loops can be derived from the set of fundamental Mandelstam identities involving only N number of loops in terms of prepotentials (as given in (6.44) for SU(3) example).

6.6 Summary and Discussion

We have defined and constructed irreducible prepotential operator for lattice gauge theory with arbitrary SU(N) gauge group. They are defined at each lattice site. The original variables like electric fields and link variables are written in terms of the prepotentials. The link operator now breaks into two parts which transform locally under SU(N) located at two ends. The link between the left and right part of the link operator is been carried by an additional $U(1)^{(N_1)}$ gauge transformation. The local gauge invariant states have also been constructed. The fundamental Mandelstam constraint for SU(N) gauge theory is also obtained in this formulation which involves only N loops passing through a particular site as opposed to the Migdal's infinite set of identities involving any arbitrary ($> N$) number of loops passing through a site.

However, apart from making the loop formulation simple in terms of local loop variables, the prepotentials also act as useful tool in calculating the spectrum of the Hamiltonian. In the next chapter we demonstrate this fact by calculating the spectrum of SU(3) Hamiltonian exactly as well as analytically on a small lattice consisting of four sites.

Chapter 7

Single Plaquette Problem and the Prepotentials

In this chapter we will illustrate that the prepotential formulation, developed in the previous part of the thesis, also serve as useful tool in calculating the spectrum of Hamiltonian lattice gauge theory. We have already seen that, the new formulation makes the loop states local and finite in number. We can also solve Mandelstam constraints using prepotentials in order to get exact orthonormal minimal loop basis. At this point it is interesting to address the issue of the spectrum of the theory exploiting prepotentials.

In the present chapter we solve the eigenvalue problem of the Hamiltonian exploiting prepotential formulation exactly as well as analytically for a small lattice consisting of only four lattice sites. Hence only one plaquette exists to contribute to the magnetic part of the Hamiltonian. Although there were earlier attepmts to solve single plaquette problem exactly, the introduction of prepotentials for non abelian gauge theories makes it much more simple compared to earlier works [47] and yields the same spectrum.

7.1 SU(2) Single Plaquette Hamiltonian

The Hamiltonian for a lattice (2.1) consisting of only one plaquette (four sites and four links) for SU(2) Lattice gauge theory can also be written as:

$$H = \frac{1}{\kappa}\sum_{l=1}^{4} E_l^2 + \kappa\,(2 - \mathrm{Tr}\,U_{plaquette}) \tag{7.1}$$

where, $\kappa(\sim 1/g^2)$ is the inverse coupling of the theory. The electric part of this Hamiltonian can be diagonalized trivially by going to the angular momentum basis. In fact, this is done in the strong coupling expansion, where contribution from the magnetic term of the Hamiltonian can be computed perturbatively. In this section we diagonalize the magnetic term instead.

Let us consider the magnetic part of the Hamiltonian. The magnetic part comprises of the link operator U's, defined on each of the four links. These link operators are basically SU(2) matrix valued operators given by:

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \tag{7.2}$$

Product of four such U's on four of the links is again an SU(2) matrix U, the trace of which gives $\mathrm{Tr}U_{plaquette} \equiv \chi(U)$, where, $\chi(U)$ is the SU(2) character. Note that, all the $U_{\alpha\beta}$'s commute among themselves (2.25). Hence $U_{\alpha\beta}$ for all values of α and β must have common eigenstates. Let us denote that common eigenstate as $|z\rangle$, such that,

$$U_{\alpha\beta}|z\rangle = z_{\alpha\beta}|z\rangle.$$

Now the eigenvalue of the full link operator U with eigenstate $|z\rangle$ is $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$. As $U^\dagger U = UU^\dagger = 1$ & det $U = 1$ is true, we must have

$$z^\dagger z = zz^\dagger = 1 \quad \& \quad \det z = 1.$$

Hence z is also an SU(2) valued matrix. The most general form of z is therefore given by: $z = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$, with z_1, z_2 satisfying $|z_1|^2 + |z_2|^2 = 1$.

Hence the eigenvalue of $\mathrm{Tr}U_{plaquette}$ will be $z_1 + z_1^*$. We now choose a compatible parametrization of z_1, z_2 as follows [48]:

$$z_1 = \cos\theta\, e^{i\frac{\omega}{2}} \quad , \quad z_2 = \sin\theta\, e^{i\frac{\xi}{2}} \tag{7.3}$$

such that,

$$z = \begin{pmatrix} \cos\theta\, e^{i\frac{\omega}{2}} & \sin\theta\, e^{i\frac{\xi}{2}} \\ -\sin\theta\, e^{-i\frac{\xi}{2}} & \cos\theta\, e^{-i\frac{\omega}{2}} \end{pmatrix} \tag{7.4}$$

where $0 \le \theta \le \dfrac{\pi}{2}$ and $0 \le \omega, \xi \le 4\pi$.

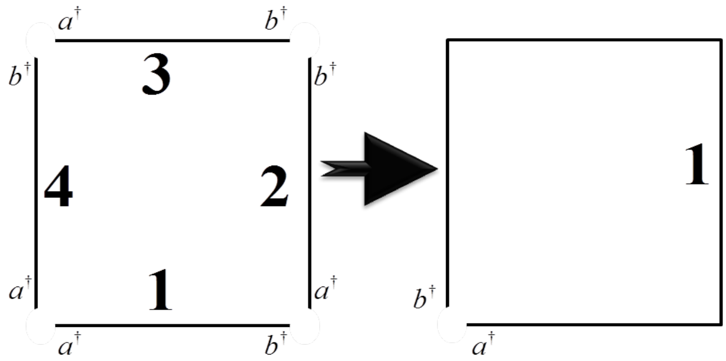

Figure 7.1: New link variable which start and end at the same point and surrounds the lattice. Prepotentials are attached at the staring and end points of each link.

Now, without loosing any generality we can always make a similarity transformation such that z becomes diagonal as follows:

$$z = \begin{pmatrix} e^{i\frac{\omega}{2}} & 0 \\ 0 & e^{-i\frac{\omega}{2}} \end{pmatrix} \quad \& \quad z_1 + z_1^* = 2\cos\frac{\omega}{2} \tag{7.5}$$

Hence $\chi(U)|z\rangle = 2\cos\frac{\omega}{2}|z\rangle$ enables us to consider $|z\rangle \equiv |\omega\rangle$.

7.1.1 Introducing Prepotentials

Introducing prepotentials for lattice gauge theory, (here the theory has SU(2) gauge invariance) each link is associated with a pair of prepotential doublet at each end (as in chapter 2). At this point we redefine the link variables by a canonical transformation from the usual ones. Instead of four links surrounding the plaquette, we consider only a single link which starts from the lower-left corner of the plaquette, surrounds it and end at the starting point itself as shown in figure 7.1. Thus the new link is basically,

$$U = U_1 U_2 U_3^\dagger U_4^\dagger \tag{7.6}$$

Now we can attach prepotentials at the ends of this new link variable. This new link has left electric field attached to it at its starting point and right electric field (parallel transport of the left electric field by the full plaquette) at its end point. Gauss law is satisfied at the origin or the lower left site. The SU(2) single plaquette Hamiltonian in terms of these new variables is:

$$H = \frac{4}{\kappa}E^2 + \kappa\left(\chi(e) - \chi(U)\right) \tag{7.7}$$

where, e is the identity element of SU(2).

Let us now consider the particular link, surrounding the plaquette. It has associated prepotentials $a^\dagger_\alpha$ and $b^\dagger_\alpha$ at it's both ends which are at the same site. Using the prepotential decomposition of the link operator developed in the chapter 2 in (2.20):

$$\chi(U) = \mathrm{T}r\hat{U} = \mathrm{T}r(\hat{U}_+ + \hat{U}_-) \tag{7.8}$$

with $\mathrm{T}r\hat{U}_+ = \frac{1}{\sqrt{(N+1)}}k_+\frac{1}{\sqrt{(N+1)}}$ and $\mathrm{T}r\hat{U}_- = \frac{1}{\sqrt{(N+1)}}k_-\frac{1}{\sqrt{(N+1)}}$, where, $k_+ = a^\dagger \cdot \tilde{b}^\dagger$, $k_- = \tilde{a}\cdot b$ and $N = a^\dagger \cdot a$ or $b^\dagger \cdot b$ having the same eigenvalue n.

Hence the normalized gauge invariant or loop state associated with this plaquette is:

$$|n\rangle = \frac{\left(a^\dagger \cdot \tilde{b}^\dagger\right)^n}{\sqrt{n!(n+1)!}}|0\rangle \equiv \frac{(k_+)^n}{\sqrt{n!(n+1)!}}|0\rangle \tag{7.9}$$

Now consider the state $|\omega\rangle$ which is the eigen state of magnetic part of the Hamiltonian, i.e

$$\mathrm{T}r\hat{U}|\omega\rangle = 2\cos\frac{\omega}{2}|\omega\rangle. \tag{7.10}$$

$|\omega\rangle$ is trivially the eigenstate $\chi(e)$ with eigenvalue 2. Since $|\omega\rangle$ is an eigenstate of $\mathrm{T}r\hat{U}$, it must be a loop state. Being gauge invariant this loop state can be written as a linear combination of all possible gauge invariant states given in (7.9) as,

$$|\omega\rangle = \sum_{n=0}^{\infty} F_n(\omega)|n\rangle \tag{7.11}$$

One can show that,

$$\begin{aligned} \left(\frac{1}{\sqrt{N+1}}k_+\frac{1}{\sqrt{N+1}}\right)|n\rangle &= |n+1\rangle \\ \& \qquad \left(\frac{1}{\sqrt{N+1}}k_-\frac{1}{\sqrt{N+1}}\right)|n\rangle &= |n-1\rangle \end{aligned} \tag{7.12}$$

Note that, the k_+, k_- and k_0 are SU(2) invariant and satisfy Sp(2,R) algebra (5.25). Hence, from (7.10) and (7.11) using (7.12) we get,

$$\sum_{n=0}^{\infty} F_n(\omega)|n+1\rangle + \sum_{n=1}^{\infty} F_n(\omega)|n-1\rangle = 2\cos\frac{\omega}{2}\sum_{n=0}^{\infty} F_n(\omega)|n\rangle \tag{7.13}$$

The equation (7.13), with the fact that $F_{-1}(\omega) = 0$ yields the recurrance relation:

$$F_{n+1}(\omega) + F_{n-1}(\omega) = 2\cos\frac{\omega}{2} F_n(\omega) \tag{7.14}$$

Solving this relation with the boundary condition $F_0(\omega) = 1$, gives the function as:

$$F_n(\omega) = \frac{\sin(n+1)\frac{\omega}{2}}{\sin\frac{\omega}{2}} \tag{7.15}$$

One can easily check that (7.15) is compatible with $F_{-1}(\omega) = 0$. Note that, SU(2) characters have the same expression as $F_n(\omega)$ with $n = 2j$, i.e,

$$F_n(\omega) \equiv \chi_{j=n/2}(\omega) = \frac{\sin(2j+1)\frac{\omega}{2}}{\sin\frac{\omega}{2}}. \tag{7.16}$$

Hence we can identify the coefficients $F_n(\omega)$ with SU(2) characters. [1]

7.1.2 SU(2) characters: Some important relations and its consequence to $|\omega\rangle$

At this point it is usefull to note some important properties of SU(2) characters as given in [38] which will be essential, and will also make the calculation of single plaquette spectrum enormously simple.

The orthogonality relation of the SU(2) character $\chi_j(\omega)$ [38],

$$\int_0^{4\pi} d\mu(\omega)\, \chi_j(\omega)\chi_{j'}(\omega) = 2\pi\delta_{jj'} \tag{7.18}$$

[1] We can also check explicitly that $F_n(\omega) \approx \sin(n+1)\frac{\omega}{2}$ satisfies (7.14).

$$\begin{aligned}
\sin(n+2)\frac{\omega}{2} + \sin n\frac{\omega}{2} &= \frac{e^{i(n+2)\frac{\omega}{2}} - e^{-i(n+2)\frac{\omega}{2}} + e^{in\frac{\omega}{2}} - e^{-in\frac{\omega}{2}}}{2i} \\
&= \frac{1}{2i}\left[e^{i(n+1)\frac{\omega}{2}}\left(e^{i\frac{\omega}{2}} + e^{-i\frac{\omega}{2}}\right) - e^{-i(n+1)\frac{\omega}{2}}\left(e^{i\frac{\omega}{2}} + e^{-i\frac{\omega}{2}}\right)\right] \\
&= 2\cos\frac{\omega}{2}\sin(n+1)\frac{\omega}{2}.
\end{aligned} \tag{7.17}$$

implies the completeness property of $|\omega\rangle$ defined in (7.11) as:

$$\begin{aligned}\int_0^{4\pi} d\mu(\omega)\,|\omega\rangle\langle\omega| &= \sum_{j,j'}\int d\mu(\omega)\chi_j(\omega)\chi_{j'}(\omega)|j\rangle\langle j'| \\ &= 2\pi\sum_j |j\rangle\langle j| = 2\pi\mathbb{I} \\ \Rightarrow\quad \frac{1}{2\pi}\int_0^{4\pi} d\mu(\omega)\,|\omega\rangle\langle\omega| &= \mathbb{I} \qquad (7.19)\end{aligned}$$

where, $d\mu(\omega) = \sin^2\frac{\omega}{2}d\omega$ is the invariant measure for SU(2). Now, for any given function $\phi_n(x)$, satisfying the completeness relation

$$\int_a^b dx\,\phi_n(x)\phi_m(x) = \delta_{nm},$$

there exists the expansion of delta function $\delta(x-t)$ in terms of these functions as:

$$\delta(x-t) = \sum_{n=0}^{\infty}\phi_n(x)\phi_n(t).$$

Using these above relations for our coefficient $F_n(\omega) \equiv \chi_j(\omega)$, we get:

$$\begin{aligned}\int_0^{4\pi}\sin^2\frac{\omega}{2}d\omega\chi_j(\omega)\chi_{j'}(\omega) &= 2\pi\delta_{jj'} \\ \text{or,}\quad \int_0^{4\pi} d\omega\frac{\sin(2j+1)\frac{\omega}{2}}{\sqrt{2\pi}}\frac{\sin(2j+1)\frac{\omega}{2}}{\sqrt{2\pi}} &= \delta_{jj'} \\ \Rightarrow\quad \sum_j\frac{\sin(2j+1)\frac{\omega}{2}}{\sqrt{2\pi}}\frac{\sin(2j+1)\frac{\omega'}{2}}{\sqrt{2\pi}} &= \delta(\omega-\omega') \\ \Rightarrow\quad \sum_j\chi_j(\omega)\chi_j(\omega') &= \frac{2\pi\delta(\omega-\omega')}{\sin^2\frac{\omega}{2}} \qquad (7.20)\end{aligned}$$

In the above derivation we have used the delta-function representation by orthogonal functions with the property that, for any two orthogonal functions $\varphi_n(x)$ and $\varphi_m(x)$:

$$\int_a^b dx\,\varphi_m(x)\varphi_n(x) = \delta_{mn} \Rightarrow \delta(x-t) = \sum_{n=0}^{\infty}\varphi_n(x)\varphi_n(t). \qquad (7.21)$$

Hence, orthonormality of the state $|\omega\rangle$ is obtained as:

$$\begin{aligned}\langle\omega|\omega'\rangle &= \sum_{j,j'}\chi_j(\omega)\chi_{j'}(\omega)\langle j|j'\rangle \\ &= \sum_j\chi_j(\omega)\chi_j(\omega') = \frac{2\pi\delta(\omega-\omega')}{\sin^2\frac{\omega}{2}} \qquad (7.22)\end{aligned}$$

7.1.3 Spectrum of the Hamiltonian

With all the calculational tool mentioned in the last section, we are now in a proper platform to calculate the energy eigenstates of the full single plaquette Hamiltonian exactly. Consider the eigenstate of the Hamiltonian H in (7.1) to be $|\epsilon\rangle$ with eigen value ϵ. i.e

$$\begin{aligned} H|\epsilon\rangle &= \epsilon|\epsilon\rangle \\ \text{or,} \quad \langle\omega|H|\epsilon\rangle &= \epsilon\langle\omega|\epsilon\rangle \end{aligned} \tag{7.23}$$

Now we define the wavefunction of single plaquette Hamiltonian as a function of the continuous parameter ω as ,

$$\langle\omega|\epsilon\rangle = \psi_\epsilon(\omega) \tag{7.24}$$

such that,

$$|\epsilon\rangle = \frac{1}{2\pi}\int_{\omega=0}^{4\pi} d\mu(\omega)\,\psi_\epsilon(\omega)|\omega\rangle \tag{7.25}$$

It readily implies,

$$\begin{aligned} \langle\omega'|\epsilon\rangle &= \frac{1}{2\pi}\int_{\omega=0}^{4\pi} d\mu(\omega)\,\psi_\epsilon(\omega)\langle\omega'|\omega\rangle \\ &= \frac{1}{2\pi}\int_{\omega=0}^{4\pi} d\mu(\omega)\,\psi_\epsilon(\omega)\frac{2\pi\delta(\omega-\omega')}{\sin^2\frac{\omega}{2}} \\ &= \psi_\epsilon(\omega') \end{aligned} \tag{7.26}$$

The action of the electric part of the Hamiltonian H_{el} on the eigenstate $|\omega\rangle$ of magnetic part of the Hamiltonian is as follows:

$$\frac{4}{\kappa}E^2|\omega\rangle = \frac{4}{\kappa}\sum_j \chi_j(\omega)j(j+1)|j\rangle \tag{7.27}$$

The SU(2) character $\chi_j(\omega)$ satisfies the relation [38]:

$$\frac{d^2\chi_j(\omega)}{d\omega^2} + \cot\frac{\omega}{2}\frac{d\chi_j(\omega)}{d\omega} + j(j+1)\chi_j(\omega) = 0 \tag{7.28}$$

Hence, the action of H_{el} on the state $|\omega\rangle$ can be written as:

$$H_{\text{el}}|\omega\rangle = -\frac{4}{\kappa}\left[\frac{d^2}{d\omega^2} + \cot\frac{\omega}{2}\frac{d}{d\omega}\right]|\omega\rangle \tag{7.29}$$

$$\Rightarrow \langle\epsilon|H_{\text{el}}|\omega\rangle^* = -\frac{4}{\kappa}\left[\frac{d^2}{d\omega^2} + \cot\frac{\omega}{2}\frac{d}{d\omega}\right]\psi_\epsilon(\omega) \tag{7.30}$$

Now, consider the eigen value equation:

$$\langle \omega | H | \epsilon \rangle = \epsilon \langle \omega | \epsilon \rangle \tag{7.31}$$

In the continuum representation the above equation becomes:

$$\begin{aligned} -\frac{4}{\kappa}\left[\frac{d^2}{d\omega^2} + \cot\frac{\omega}{2}\frac{d}{d\omega}\right]\psi_\epsilon(\omega) + 2\kappa\left(1-\cos\frac{\omega}{2}\right)\psi_\epsilon(\omega) = \epsilon\psi_\epsilon(\omega) \\ \text{or,}\quad \left[\sin\frac{\omega}{2}\frac{d^2}{d\omega^2} + \cos\frac{\omega}{2}\frac{d}{d\omega}\right]\psi_\epsilon(\omega) + \frac{\kappa}{4}\left[\epsilon - 2\kappa\left(1-\cos\frac{\omega}{2}\right)\right]\sin\frac{\omega}{2}\psi_\epsilon(\omega) = 0 \end{aligned} \tag{7.32}$$

This is precisely the damped Mathieu equation. Now, we define,

$$\phi_\epsilon(\omega) = \sin\frac{\omega}{2}\psi_\epsilon(\omega) \tag{7.33}$$

such that, the above equion becomes,

$$\left[\frac{d^2}{d\omega^2} + \frac{1}{4}\right]\phi_\epsilon(\omega) + \frac{\kappa}{4}\left[\epsilon - 2\kappa\left(1-\cos\frac{\omega}{2}\right)\right]\phi_\epsilon(\omega) = 0 \tag{7.34}$$

This is now the Mathieu's equation. From the properties of SU(2) character $\chi_j(-\omega) = \chi_j(\omega)$ and $\chi_j(\omega + 4\pi) = \chi_j(\omega)$, it is assured that the wavefunction will have the following symmetry properties:

$$\phi_\epsilon(-\omega) = \phi_\epsilon(\omega) \quad \& \quad \phi_\epsilon(\omega + 4\pi) = \phi_\epsilon(\omega) \tag{7.35}$$

Hence the wavefunction must be an even periodic solution of Mathieu equation. Comparing (7.34) with the standard Mathieu equation $y'' + (a - 2q\cos 2z)y = 0$ given in [49] we get the solution of the equation as the even Mathieu function 'MathieuC[a, q, z]' in Mathematica which is a cosine elliptic function with the following argument:

$$a = -4(-1 + 2\kappa^2 - \kappa\epsilon) \tag{7.36}$$

$$q = -4\kappa^2 \tag{7.37}$$

$$z = \frac{\omega}{4} \tag{7.38}$$

Now, we have the following observations:

- Our wavefunction is a function of a single continuous parameter ω with the symmetry $\omega \to \omega + 4\pi$. But from the Mathieu function (y) we see that the solution of the Schrödinger's equation ($\phi(\omega) \equiv y(z)$) is function of $z = \omega/4$. Hence, the wavefunction is also a function of $\omega' = \omega/4$ with period π.

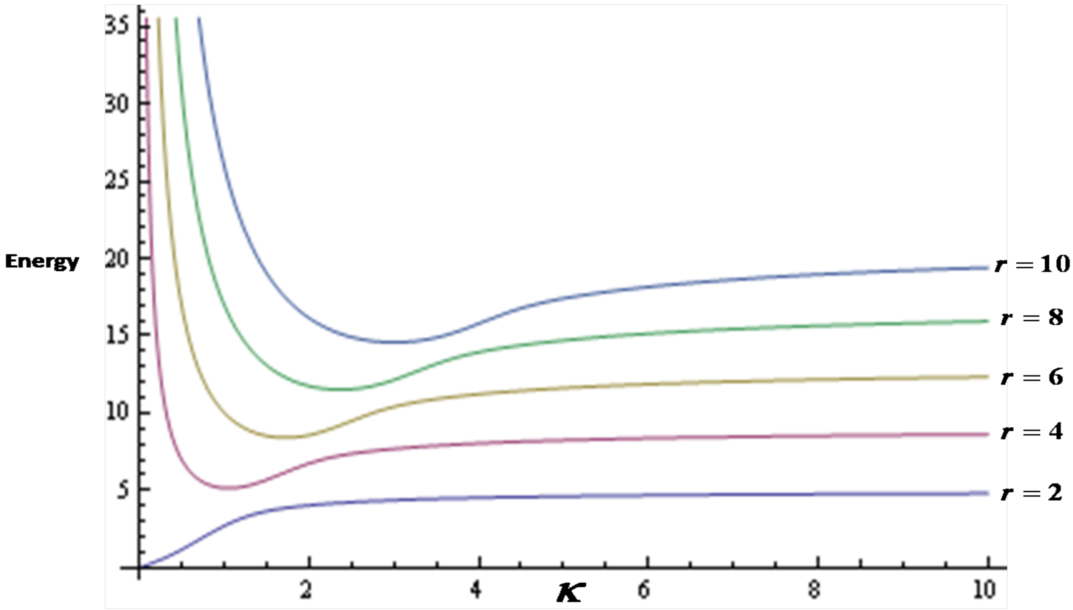

Figure 7.2: Lowest five energy levels of the Single palquette Hamiltonian. The X axis is the coupling κ and Y axis is energy. The Energy levels show strong coupling behavior for small values of κ and for large κ, i.e in the weak coupling regime ($\kappa \rightarrow \infty$) all energy levels show finite mass gaps.

- The coupling of the Hamiltonian κ is related to the parameter q of the Mathieu equation as shown in (7.37).
- The Mathieu Functions are solution of the Mathieu equation with only some allowed values of the parameter a, known as the characteristic coefficient of Mathieu equation. Different characteristic coefficients denotes different discrete solutions of Mathieu equation. As we see in (7.36), here the characteristic coefficients are related to the energy ϵ of the single plaquette Hamiltonian. Hence, different allowed values of the discrete characteristic coefficient of Mathieu equation a_r with $r = 2, 4, ...$ implies discrete energy levels ϵ_r of the single plaquette Hamiltonian. The functional relationship of the characteristic coefficients with the parameter q given in [49], gives the dependence of the enrgy spectrum to the coupling according to (7.36) as shown in figure 7.2. *The energy spectrum shows non-zero mass-gap as expected.*

In the next section we consider the single plaquette problem for the gauge group SU(3) as well as arbitrary SU(N).

7.2 Single Plaquette in SU(3) Lattice Gauge Theory

In this section we will generalize the above method to calculate the spectrum of single plaquette Hamiltonian for arbitrary SU(N) gauge group. To start with, let us consider the example of SU(3) gauge theory defined on a lattice consisting of only four sites. As discussed in the case of SU(2), we redefine the link variables such that only one relevant link exists. Hence one can construct gauge invariant loops for the theory only out of this link. In other words, the plaquette is consructed out of only one link, carrying the link variable, which is an operator valued matrix. Likewise in SU(2) case, we can always make a similarity transformation such that the SU(3) matrix $\mathcal{U}$ becomes diagonal. Let us consider the SU(3) link operator matrix representing the full plaquette to be a diagonal one. As we also know that, all the elements of the link operator matrix do commute with each other resulting a common eigen state $|z\rangle$ for the full U matrix. i.e,

$$U_{\alpha\beta}|z\rangle = z_{\alpha\beta}|z\rangle.$$

Hence the eigenvalue matrix z must also be a SU(3) matrix one given as:

$$z \equiv \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_1} & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)} \end{pmatrix} \tag{7.39}$$

satisfying,

$$\mathrm{Tr}\, U_{plaqquette}|z\rangle = \mathrm{Tr}\, z|z\rangle \equiv \left(e^{i\theta_1} + e^{i\theta_2} + e^{-i(\theta_1+\theta_2)}\right)|z\rangle. \tag{7.40}$$

Hence, we can consider $|z\rangle \equiv |\theta_1, \theta_2\rangle$.

The Hamiltonian for SU(3) single plaquette is given by,

$$H = \underbrace{\frac{1}{\kappa}\sum_{l=1}^{4} E_l^2}_{H_{el}} + \kappa \underbrace{\left(3 - \frac{1}{2}\mathrm{Tr}U_{plaquette} - \frac{1}{2}\mathrm{Tr}U^{\dagger}_{plaquette}\right)}_{H_{mag}}. \tag{7.41}$$

Now, the eigenstates of Tr$U_{plaquette}$ will automatically be eigenstates of H_{mag} with the following eigen value:

$$\begin{aligned} H_{mag}|\theta_1,\theta_2\rangle &= \left[3-\frac{1}{2}(e^{i\theta_1}+e^{i\theta_2}+e^{-i(\theta_1+\theta_2)})-\frac{1}{2}(e^{-i\theta_1}+e^{-i\theta_2}+e^{i(\theta_1+\theta_2)})\right]|\theta_1,\theta_2\rangle \\ &= 3-(cos\,\theta_1+cos\,\theta_2+cos\,(\theta_1+\theta_2))|\theta_1,\theta_2\rangle \end{aligned} \quad (7.42)$$

7.2.1 Introducing Prepotentials and Treating H_{mag}

Likewise SU(2) case, we now attach a set of two ($A^{\dagger\alpha}(L/R) \in 3$ and $B^\dagger_\alpha(L/R) \in 3^*$) prepotential triplets and anti-triplets (since we are working with SU(3) gauge group) to both the ends (L/R) of the link. Since, the link we consider here, surrounds the plaquette, both the left and right prepotentials are situated at the same point. Thus the gauge invariant states constructed out of prepotentials are:

$$\begin{aligned} |p,q\rangle \equiv |p,q\rangle_L \otimes |q,p\rangle_R &= \mathcal{N}_{p,q}\left(A^\dagger(L)\cdot B^\dagger(R)\right)^p\left(A^\dagger(R)\cdot B^\dagger(L)\right)^q|0\rangle \\ &= \mathcal{N}_{p,q}\left(a^\dagger(L)\cdot b^\dagger(R)\right)^p\left(a^\dagger(R)\cdot b^\dagger(L)\right)^q|0\rangle \end{aligned} \quad (7.43)$$

Where, $\mathcal{N}_{p,q}$ is normalization factor. Moreover. the eigenstates of Tr $U_{plaquette}$ are gauge invariant and hence must be a linear combination of all possible gauge invariant states $|p,q\rangle$ constructed in terms of prepotentials. Hence it must be of the form,

$$|\theta_1,\theta_2\rangle = \sum_{p,q=0}^{\infty} F_{p,q}(\theta_1,\theta_2)|p,q\rangle. \quad (7.44)$$

Again, as derived in (5.57), the link operator consists of three terms:

$$U \equiv U^+_+ + U^-_- + U^{+-}_{-+} \quad (7.45)$$

analogous to (7.8) for SU(2) case. Similar to the SU(2) analysis done before, it can also be shown that,

$$\begin{aligned} U^+_+|p,q\rangle &= |p+1,q\rangle \\ U^-_-|p,q\rangle &= |p,q-1\rangle \\ U^{+-}_{-+}|p,q\rangle &= |p-1,q+1\rangle \end{aligned} \quad (7.46)$$

Hence for $|\theta_1, \theta_2\rangle$ to be the eigenstate of $\mathrm{Tr}\, U_{plaquette}$ as in (7.40), we get the following relation to be satisfied by the coefficient $F_{p,q}(\theta_1, \theta_2)$ in (7.44):

$$\begin{aligned} & F_{p-1,q}(\theta_1,\theta_2) + F_{p,q+1}(\theta_1,\theta_2) + F_{p+1,q-1}(\theta_1,\theta_2) \\ = & \left(e^{i\theta_1} + e^{i\theta_2} + e^{-i(\theta_1+\theta_2)}\right) F_{p,q}(\theta_1,\theta_2). \end{aligned} \tag{7.47}$$

Similar to the SU(2) case, we observe that, the SU(3) character functions [50] satisfy the above recurrence relation exactly. This can be shown explicitly by considering the following functional form of SU(3) character [50]:

$$\begin{aligned} \chi_{p,q}(\theta_1,\theta_2) = & -\frac{i}{S(\theta_1,\theta_2)} \Bigg[\exp(ip\theta_1 - iq\theta_2) - \exp(-iq\theta_1 + ip\theta_2) \\ & + \exp(-ip(\theta_1+\theta_2)\left(\exp(-iq\theta_1) - \exp(-iq\theta_2)\right) \\ & + \exp(iq(\theta_1+\theta_2)\left(\exp(ip\theta_2) - \exp(ip\theta_1)\right) \Bigg] \end{aligned} \tag{7.48}$$

where,

$$S(\theta_1,\theta_2) = 8sin\left(\frac{\theta_1-\theta_2}{2}\right) sin\left(\frac{\theta_1+2\theta_2}{2}\right) sin\left(\frac{2\theta_1+\theta_2}{2}\right) \tag{7.49}$$

Hence we can conclude that, the eigenstates of H_{mag} are,

$$|\theta_1,\theta_2\rangle = \sum_{p,q=0}^{\infty} \chi_{p,q}(\theta_1,\theta_2)|p,q\rangle. \tag{7.50}$$

7.2.2 Treating H_{el}

We now consider the electric part of the single plaquette Hamiltonian given by, $H_{el} = E^2$.

The loop states $|p,q\rangle$ are eigenstates of the electric part of the Hamiltonian as that is the Casimir operator with the following eigenvalue (5.38):

$$E^2|p,q\rangle = \frac{1}{3}\left(p^2 + q^2 + 3p + 3q + pq\right)|p,q\rangle. \tag{7.51}$$

Hence, it implies,

$$H_{el}|\theta_1,\theta_2\rangle = \sum_{p,q=0}^{\infty} \chi_{p,q}(\theta_1,\theta_2)\frac{1}{3}\left(p^2 + q^2 + 3p + 3q + pq\right)|p,q\rangle. \tag{7.52}$$

At this point it is important to explore some properties of SU(3) character which are necessary for our purpose.

Characters have the following symmetry properties which can be checked explicitly from (7.48):

$$\chi^*_{p,q}(\theta_1,\theta_2) = \chi_{q,p}(\theta_1,\theta_2) \;\; \& \;\; \chi_{p,q}(\theta_1,\theta_2) = -\chi_{p,q}(\theta_2,\theta_1). \tag{7.53}$$

The orthonormality of the characters are given as:

$$\int_{-\pi}^{\pi} d\mu(\theta_1,\theta_2)\chi^*_{p,q}(\theta_1,\theta_2)\chi_{p',q'}(\theta_1,\theta_2) = (2\pi)^2\delta_{pp'}\delta_{qq'} \tag{7.54}$$

where, $d\mu$ is the invariant measure on SU(3) group manifold given by,

$$d\mu(\theta_1,\theta_2) = S^2(\theta_1,\theta_2)d\theta_1 d\theta_2, \qquad -\pi \leq \theta_1,\theta_2 \leq \pi \tag{7.55}$$

with $S(\theta_1,\theta_2)$ given in (7.49). Using (7.54), we find:

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_{-\partial}^{\pi} d\mu(\theta_1,\theta_2)|\theta_1,\theta_2\rangle\langle\theta_1,\theta_2| &= \sum_{p,q,p',q'}\int_{-\pi}^{\pi} d\mu(\theta_1,\theta_2)\chi^*_{p,q}(\theta_1,\theta_2)\chi_{p',q'}(\theta_1,\theta_2)|p',q'\rangle\langle p,q| \\ &= \sum_{p,q}|p,q\rangle\langle p,q| = \mathbb{I} \end{aligned}\tag{7.56}$$

Further one can also derive the relation:

$$\sum_{pq}\chi^*_{p,q}(\theta_1,\theta_2)\chi_{p,q}(\theta'_1,\theta'_2) = \frac{(2\pi)^2}{S^2(\theta_1,\theta_2)}\delta(\theta_1-\theta'_1)\delta(\theta_2-\theta'_2) \tag{7.57}$$

following the same logic used to derive (7.20) in the case of SU(2) characters. This relation eventually leads to the orthonormality of the states $|\theta_1,\theta_2\rangle$ as:

$$\begin{aligned}\langle\theta_1,\theta_2|\theta'_1,\theta'_2\rangle &= \sum_{p,q,p',q'}\chi^*_{p,q}(\theta_1,\theta_2)\chi_{p',q'}(\theta'_1,\theta'_2)\langle p,q|p',q'\rangle \\ &= \sum_{pq}\chi^*_{p,q}(\theta_1,\theta_2)\chi_{p,q}(\theta'_1,\theta'_2) = \frac{(2\pi)^2}{S^2(\theta_1,\theta_2)}\delta(\theta_1-\theta'_1)\delta(\theta_2-\theta'_2)\end{aligned} \tag{7.58}$$

Now, the most important property of SU(3) (as well as of arbitrary SU(N)) character functions is that, they are eigenfunctions of the Laplace-Beltrami operator defined on group manifold, and the corresponding eigenvalue is the quadratic Casimir operator's eigenvalue for that representation. Now, note from (7.52), that

$$H_{el}|\theta_1,\theta_2\rangle = \sum_{p,q}\left[-\nabla^2\chi_{p,q}(\theta_1,\theta_2)|p,q\rangle\right] = -\nabla^2|\theta_1,\theta_2\rangle. \tag{7.59}$$

7.2.3 Treating full Hamiltonian

Having analyzed the action of electric and magnetic part separately on the continuum states $|\theta_1, \theta_2\rangle$ for SU(3) single plaquette problem, we are now going to look at the full Hamiltonian given by,

$$H = \frac{4}{\kappa} H_{el} + \kappa H_{mag}. \tag{7.60}$$

We already know that,

$$H|\theta_1, \theta_2\rangle = \left[-\frac{4}{\kappa} \nabla^2 + \kappa \left(3 - cos\theta_1 - cos\theta_2 - cos(\theta_1 + \theta_2) \right) \right] |\theta_1, \theta_2\rangle. \tag{7.61}$$

Hence, the eigenvalue equation for single plaquette Hamiltonian of SU(3),

$$H|\epsilon\rangle = \epsilon|\epsilon\rangle \quad \Rightarrow \quad \langle\theta_1, \theta_2|H|\epsilon\rangle = \epsilon\langle\theta_1, \theta_2|\epsilon\rangle \tag{7.62}$$

takes the continuum representation,

$$\left[-\frac{4}{\kappa} \nabla^2 + \kappa \left(3 - cos\theta_1 - cos\theta_2 - cos(\theta_1 + \theta_2) \right) \right] \psi_\epsilon(\theta_1, \theta_2) = \epsilon \psi_\epsilon(\theta_1, \theta_2). \tag{7.63}$$

where, we have exploited the relations:

$$|\epsilon\rangle = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\mu(\theta_1, \theta_2) \psi_\epsilon(\theta_1, \theta_2) |\theta_1, \theta_2\rangle \tag{7.64}$$

with $\psi_\epsilon(\theta_1, \theta_2) = \langle\theta_1, \theta_2|\epsilon\rangle$.

The action of Laplace-Beltrami operator on some SU(3) invariant function is as follows [51, 52]:

$$\nabla^2 f(\theta_1, \theta_2) = \sum_{i=1}^{3} \frac{1}{S^2} \frac{\partial}{\partial\theta_i} S^2 \frac{\partial}{\partial\theta_i} f(\theta_1, \theta_2) \tag{7.65}$$

where, S is the socalled van der Monde's determinat given in (7.49)[2] and $\theta_3 = -(\theta_1 + \theta_2)$.

[2]Note that, for SU(2) case, $S(\omega) = sin\frac{\omega}{2}$, and hence,

$$\nabla^2 = \frac{1}{sin^2\frac{\omega}{2}} \frac{d}{d\omega} sin^2\frac{\omega}{2} \frac{d}{d\omega} = \frac{d^2}{d\omega^2} + cot\frac{\omega}{2} \frac{d}{d\omega}.$$

Hence, we can identify (7.29) as

$$H_{el}|\omega\rangle = -\frac{1}{\kappa} \nabla^2 |\omega\rangle.$$

Replacing θ_3, and after doing some algebra we get,

$$\nabla^2 f = \frac{2}{S}\frac{\partial^2}{\partial\theta_1^2}(Sf) + \frac{2}{S}\frac{\partial^2}{\partial\theta_2^2}(Sf) + \frac{2}{S}\frac{\partial^2}{\partial\theta_1\partial\theta_2}(Sf) - \frac{2f}{S^2}\left(\frac{\partial^2 S}{\partial\theta_1^2} + \frac{\partial^2 S}{\partial\theta_2^2} + \frac{\partial^2 S}{\partial\theta_1\partial\theta_2}\right) \quad (7.66)$$

Using this in (7.63), we get the single plaquette Schrödinger's equation for SU(3) as:

$$\Bigg[-\frac{4}{\kappa}\left[\frac{2}{S}\frac{\partial^2}{\partial\theta_1^2}S + \frac{2}{S}\frac{\partial^2}{\partial\theta_2^2}S + \frac{2}{S}\frac{\partial^2}{\partial\theta_1\partial\theta_2}S - \frac{2}{S^2}\left(\frac{\partial^2 S}{\partial\theta_1^2} + \frac{\partial^2 S}{\partial\theta_2^2} + \frac{\partial^2 S}{\partial\theta_1\partial\theta_2}\right)\right]$$
$$+\kappa\left(3 - cos\theta_1 - cos\theta_2 - cos(\theta_1+\theta_2)\right)\Bigg]\psi_\epsilon(\theta_1,\theta_2) = \epsilon\psi_\epsilon(\theta_1,\theta_2), \quad (7.67)$$

or equivalently,

$$\Bigg[-\left[\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2} + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\right] + \frac{\kappa^2}{8}\Bigg[3 - \frac{\epsilon}{\kappa} - \frac{2}{\kappa^2 S^2}\left(\frac{\partial^2 S}{\partial\theta_1^2} + \frac{\partial^2 S}{\partial\theta_2^2} + \frac{\partial^2 S}{\partial\theta_1\partial\theta_2}\right)$$
$$-\left(cos\theta_1 + cos\theta_2 + cos(\theta_1+\theta_2)\right)\Bigg]\Bigg]\phi_\epsilon(\theta_1,\theta_2) = 0 \quad (7.68)$$

where, $\phi_\epsilon(\theta_1,\theta_2) = S(\theta_1,\theta_2)\psi_\epsilon(\theta_1,\theta_2)$. Note that, the compact range of the configuration space variables ($-\pi \le \theta_1, \theta_2 \le \pi$) implies discrete spectrum of the Single plaquette Hamiltonian. However, the exact analytic solution of this elliptic equation is not available. In the next section we write the single plaquette Schrödinger's equation for the gauge group SU(N) with arbitrary N.

7.3 Generalization to arbitrary SU(N)

Just like SU(3) analysis, under suitable parametrization an arbitrary SU(N) matrix is of the form:

$$\begin{pmatrix} \exp(i\theta_1) & 0 & \dots & 0 \\ 0 & \exp(i\theta_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp -i(\theta_1+\theta_2+\ldots+\theta_{N-1}) \end{pmatrix} \quad (7.69)$$

and so is the link operator. Hence, the eigenstates of the magnetic part of the SU(N) single plaquette Hamiltonian can be characterized by $N-1$ angles, each ranging from $-\pi$

to π. i.e we have,

$$H_{mag}|\theta_1, \theta_2, \ldots, \theta_{N-1}\rangle = (N - cos\theta_1 \ldots - cos\theta_{N-1} - cos(\theta_1 + \theta_2 + .. + \theta_{N-1})) |\theta_1, \theta_2, \ldots, \theta_{N-1}\rangle. \tag{7.70}$$

The weak coupling eigen state is,

$$|\theta_1, \theta_2, \ldots, \theta_{N-1}\rangle = \sum_{n_1,\ldots,n_{N-1}} \chi_{n_1,\ldots,n_{N-1}}(\theta_1, \theta_2, \ldots, \theta_{N-1})|n_1, \ldots, n_{N-1}\rangle, \tag{7.71}$$

where, $|n_1, \ldots, n_{N-1}\rangle$ is the strong coupling eigen vector of SU(N) single plaquette lattice gauge theory. The weak coupling basis is expanded in the linear combination of strong coupling basis with coefficients, $\chi_{n_1,\ldots,n_{N-1}}(\theta_1, \theta_2, \ldots, \theta_{N-1})$, which are the character functions of SU(N). Now likewise SU(2) and SU(3) case, we exploit the properties of general SU(N) characters which tells that character functions are eigenfunctions of the Laplace-Beltrami operator with eigenvalues equal to that of the quadratic Casimir operator which is basically the electric part of the Hamiltonian. The Laplace Beltrami operator for U(N) is the differential operator of form:

$$\nabla^2 = \sum_{i=1}^{N} \frac{1}{S^2}\frac{\partial}{\partial\theta_i} S^2 \frac{\partial}{\partial\theta_i} \tag{7.72}$$

where, S is:

$$S(\theta_1, \ldots, \theta_N) = \prod_{i<j} 2sin\frac{\theta_i - \theta_j}{2} \tag{7.73}$$

To deduce the SU(N) result one needs to replace $\theta_N = -(\theta_1 + \ldots + \theta_{N-1})$ in the above expressions. The exactly similar analysis done for the case of SU(2) and SU(3) would lead to the single plaquette Schrödinger's equation as follows:

$$\sum_{i=1}^{N} \left[-\frac{1}{\kappa}\left[\frac{1}{S}\frac{\partial^2}{\partial\theta_i^2} S - \frac{1}{S}\frac{\partial^2 S}{\partial\theta_i^2}\right] + \kappa\left(1 - cos\theta_i\right)\right]\psi_\epsilon(\theta_1, \ldots, \theta_N) = \epsilon\psi_\epsilon(\theta_1, \ldots, \theta_N), \tag{7.74}$$

Note that, for the case of U(N) gauge theory [51] the above equation becomes separable in the variables θ_i, with $i = 1, N$ and each equation is a Mathieu equation as discussed for SU(2) case. However, for the case of SU(N), all the variables do not get decoupled as $\theta_N = -(\theta_1 + \ldots + \theta_{N-1})$ and hence cannot be solved exactly.

7.4 Summary

We have redefined the link variables for a single plaquette such that the magnetic part of the Hamiltonian becomes easy to handle. We associated prepotential operators to the links and could solve the problem easily using SU(N) invariant algebra. We also see from the energy spectrum that in such a crude limit the theory still shows non-zero mass-gap.

This single plaquette analysis is a crucial tool towards the weak coupling expansion of the theory. One can perform a weak coupling perturbation expansion of lattice gauge theory, where the unperturbed Hamiltonian is the sum of single plaquette Hamiltonians over all plaquettes. Note that, the full magnetic part of the Hamiltonian is then contained in the unperturbed one. The interaction between plaquettes is to be calculated via perturbation expansion. Work in this direction is going on.

Chapter 8

Summary and Future Directions

This thesis has been concerned with the study mutually independent loop states and their dynamics in SU(2) and higher SU(N) lattice gauge theories using the Hamiltonian prepotential formulation. The study has been motivated by the fact that in the continuum limit ($g^2 \to 0$) all these independent solutions will play important role in finding the spectrum of theory. Infact, unlike weak coupling limit, construction of loop states and their dynamics is extremely easy in the strong coupling ($g^2 \to \infty$) limit and has been exploited since the early days [1, 2]. However, it was shown that it is not possible [53] to reach weak coupling limit from the strong coupling regime by strong coupling perturbative correction as there exists a particular value of the coupling constant where the theory undergoes a roughening transition [53]. Hence to get the continuum limit from lattice theory one needs to perform a gauge invariant expansion in the weak coupling regime itself, which is somewhat dual to the strong coupling expansion. This weak coupling regime of lattice gauge theory in terms of loops is less explored because in this regime, all the loops of all possible shapes and sizes start contributing to the theory making the basis highly over-complete. In this thesis we have shown that the prepotential formulation enable us to construct mutually orthonormal loop states locally at every lattice site. Infact, as discussed earlier, the two major stumbling blocks in the loop approach to gauge theory were the non-locality and proliferation of independent loop states with lattice size. We have systematically developed ideas and techniques to reformulate SU(2) lattice gauge theories in loop space without any spurious loop degrees of freedom. We have solved SU(2) Mandelstam constraints leading to an orthonormal loop basis which is complete and characterized exactly by $3(d-1)$ angular momentum quantum numbers per lattice site [29]. Further, in this basis the dynamics is governed by $3nj$ coefficients of the second kind and therefore it is highly geometrical.

In the later part of the thesis we extended these SU(2) prepotential ideas to SU(3) and then general arbitrary SU(N). In chapter 5, we define and construct prepotential operator for SU(3) lattice gauge theory. We then found that, unlike SU(2), the SU(3) gauge theory Hilbert space is not same but is contained in the SU(3) prepotential Hilbert space. We also observed that this SU(3) complication is due to an old and well known group theoretical problem of multiplicity associated with SU(N) ($N \geq 3$) representations [40, 46, 54]. This motivated us to construct SU(3) irreducible prepotential operators in chapter 5. The SU(3) irreducible Schwinger bosons or prepotentials solve the multiplicity problem and make SU(3) representations as simple as SU(2) representation. We then constructed the link operators in terms of irreducible prepotentials which acting on strong coupling vacuum directly creates the states in the gauge theory Hilbert space. These link operators also satisfy unitary conditions $U^\dagger U = UU^\dagger = 1$, $\det U = +1$. Further, like in SU(2) case, in prepotential formalism the link operator matrix breaks into left and right parts, which transform entirely by the left or right gauge transformations residing at the left or right end of the link. The abelian flux lines connects these two ends of a link. We have also constructed all possible gauge invariant operators and states locally at each site of a d-dimensional lattice. In terms of prepotentials we then found out the fundamental Mandelstam identities for SU(3) locally at each site. All the Mandelstam identities known in the literature for SU(3) have also been caste in the local form using prepotentials and all of these can actually be derived from the fundamental one. In chapter 6 we generalized all the above ideas to arbitrary SU(N) gauge group. In chapter 7 we have focused on calculating the spectrum of the Hamiltonian using prepotentials. We could calculate the spectrum for SU(2) gauge theory defined on a small lattice consisting of four sites using SU(2) invariant Sp(2,R) algebra. For SU(3) and SU(N) theory we again write the single plaquette Schrödinger equation in terms of gauge invariant variables and the gauge invariant algebra using prepotential formulation. In the case of SU(2) lattice gauge theory, using the orthonormal basis in terms of angular momentum, we are now constructing a basis which diagonalizes the magnetic field term instead. The dynamics in such basis will be directly relevant in the weak coupling couninuum limit.

Appendix A

SU(N) Coherent States

The concept of coherent states originally introduced by Schrödinger [61] in the context of harmonic oscillators has been generalized the group SU(2) and have found wide range of applications [62, 65–68] like the original one [62–66].

In the simplest example of the Heisenberg-Weyl group, the Lie algebra contains three generators. It is defined in terms of creation annihilation operators $(a, a^\dagger)$ satisfying

$$[a, a^\dagger] = \mathcal{I}, \quad [a, \mathcal{I}] = 0, \quad [a^\dagger, \mathcal{I}] = 0 \ . \tag{A.1}$$

This algebra has only one infinite dimensional unitary irreducible representation. The states within this representation are the occupation number states $|n\rangle \equiv \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$ with $n = 0, 1, 2...$. The coherent states of the Heisenberg-Weyl group are defined over a complex manifold as:

$$|z\rangle_{[\infty]} \;=\; \exp(z a^\dagger)\ |0\rangle = \sum_{n=0}^{\infty} F_n(z)\ |n\rangle. \tag{A.2}$$

In (A.2) the subscript $[\infty]$ on the coherent states is the irreducible representation index. It implies that these coherent states are defined over the infinite dimensional irreducible representation of the group. The sum in (A.2) runs over all the basis vectors $|n\rangle$ belonging to this infinite dimensional representation. The coefficients:

$$F_n(z) \;=\; \frac{z^n}{\sqrt{n!}} \tag{A.3}$$

are the coherent state expansion coefficients which are analytic functions of the group manifold coordinate z. The resolution of identity property of the coherent state (A.2) follows from the group transformation property. Let us define the operator: $\mathcal{O}_{[\infty]} \equiv \int e^{-|z|^2} dz d\bar{z}\ |z\rangle_{[\infty]}\ {}_{[\infty]}\langle z|$. Under the Heisenberg Weyl group element [66] $g_{\rm hw} \equiv exp\left(i\ \alpha + wa - \bar{w}a^\dagger\right)$:

$$|z\rangle_{[\infty]} \rightarrow e^{i\ \alpha + zw - \frac{w\bar{w}}{2}} |z - \bar{w}\rangle_{[\infty]}.$$

It is trivial to see that the operator $\mathcal{O}_{[\infty]}$ defined above is invariant under $g_{\rm hw}$. Therefore, by Schurs lemma it is proportional to unity operator.

A.1 $SU(2)$ Coherent States

The Heisenberg Weyl coherent state construction can be readily generalized to the simplest compact group SU(2) by utilizing the Schwinger representation of SU(2) Lie algebra: $[J^{\rm a}, J^{\rm b}] = i\epsilon^{\rm abc} J^{\rm c}$. We define [33][1]

$$J^{\rm a}\ \equiv\ \frac{1}{2}\ a^\dagger_\alpha\ (\sigma^{\rm a})^\alpha{}_\beta\ a^\beta. \tag{A.4}$$

In (A.4) $\sigma^{\rm a}$ with a $= 1, 2, 3$ denote the three Pauli matrices. The doublet of harmonic oscillator creation and annihilation operators a^α and $a^\dagger_\alpha$ or equivalently Schwinger bosons in (A.4) satisfy the simple bosonic commutation relations $[a^\alpha, a^\dagger_\beta] = \delta^\alpha{}_\beta$ with $\alpha, \beta = 1, 2$. The vacuum state $|0, 0\rangle$ of these two oscillators will be denoted by $|0\rangle$. Under SU(2) transformations the Schwinger boson creation operators transform as doublets:

$$a^\dagger_\alpha \rightarrow a^\dagger_\beta \left(\exp i\theta^{\rm a} \frac{\sigma^{\rm a}}{2}\right)^\beta{}_\alpha. \tag{A.5}$$

The defining equations (A.4) imply that the SU(2) Casimir operator is simply the total number operator:

$$\mathcal{C} \equiv \sum_{\alpha=1}^{2} a^\dagger_\alpha a^\alpha \equiv a^\dagger \cdot a. \tag{A.6}$$

The eigenvalues of $\mathcal{C}$ will be denoted by n. The various states in the irreducible representation $n(= 2j)$ are:

$$|\alpha_1\alpha_2....\alpha_n\rangle_{[n]} \equiv a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} .....a^\dagger_{\alpha_n} |0\rangle \tag{A.7}$$

[1]This is exactly same as the SU(2) electric fields constructed out of prepotential in (2.11)

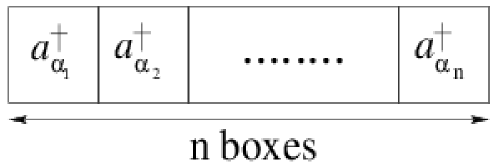

Figure A.1: SU(2) Young table in the $n = 2j$ representation. The monomial state (A.7) carries the horizontal permutation symmetries of this Young tableau.

The corresponding SU(2) Young tableau is shown in Figure (A.1). Note that the state in (A.7) is invariant under all $n!$ permutations of the SU(2) indices $\alpha_1, \alpha_2, \cdots, \alpha_n$. This is because all SU(2) creation operators on the right hand side of (A.7) commute amongst themselves. The state (A.7) is same as that in (2.16). In other words, the SU(2) Schwinger boson creation operators carry the symmetries of the SU(2) Young tableau[2] which is shown in Figure (A.1). Therefore, the $(n + 1)$ states in (A.7) belong to SU(2) irreducible representation with total angular momentum $j = \frac{n}{2}$.

The SU(2) group manifold S^3 can also be described by a doublet of of complex numbers (z^1, z^2) of unit magnitude:

$$|z|^2 \equiv |z_1|^2 + |z_2|^2 = 1. \tag{A.8}$$

This is because any SU(2) matrix $\mathcal{U}_2$ can be written as:

$$\mathcal{U}_2 = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \tag{A.9}$$

with $\mathcal{U}_2^\dagger \mathcal{U}_2 = \mathcal{U}_2 \mathcal{U}_2^\dagger = 1$, $\det(\mathcal{U}_2) = 1$. At this stage one can trivially combine the SU(2) irreducible states in (A.7) and the SU(2) group manifold coordinates in (A.8) to construct the generating function of the SU(2) coherent states:

$$|z\rangle \equiv |z^1, z^2\rangle = exp\left(z \cdot a^\dagger\right)|0\rangle = \sum_{n=0}^{\infty} \frac{\left(z \cdot a^\dagger\right)^n}{n!}|0\rangle = \sum_{n=0}^{\infty} |z\rangle_n. \tag{A.10}$$

Above $z \cdot a^\dagger \equiv z^1 a_1^\dagger + z^2 a_2^\dagger$ and $|z\rangle_{[n]}$ is the coherent state in the SU(2) representation

[2]This obvious symmetry argument will not be true for higher SU(N) (see chapter 5 and 6 as well as appendix A.2 and A.3) leading to the definition of SU(N) irreducible Schwinger bosons. In terms of SU(N) irreducible Schwinger bosons the SU(N) irreducible states will be monomials like (A.7).

$j = n/2$, hence it is written as:

$$|z\rangle_{[j]} = \sum_{\alpha_1,\cdots,\alpha_{2j}=1}^{2} F^{\alpha_1\ldots\alpha_{2j}}(z) a^{\dagger}_{\alpha_1}\cdots a^{\dagger}_{\alpha_{2j}}|0\rangle = \sum_{\alpha_1,\cdots,\alpha_{2j}=1}^{2} F^{\alpha_1\ldots\alpha_{2j}}(z) \underbrace{|\alpha_1\alpha_2\cdots\alpha_{2j}\rangle_{[j]}}_{SU(2)\ irrep.\ j=\frac{n}{2}} \tag{A.11}$$

Like in Heisenberg Weyl case (A.3), the SU(2) coherent state structure functions in the irreducible representation j are:

$$F^{\alpha_1\alpha_2\ldots\alpha_{2j}}(z^1,z^2) \equiv \frac{1}{(2j)!} z^{\alpha_1} z^{\alpha_2} \ldots z^{\alpha_{2j}}. \tag{A.12}$$

Note that they are analytic functions of group manifold coordinates. The resolution of identity property again follows from the group transformation laws. The coherent state structure in (A.10) and the the SU(2) transformations (A.5) imply that under group transformations: $|z^1,z^2\rangle_{[j]} \to |z'^1,z'^2\rangle_{[j]}$ where (z'^1,z'^2) are the SU(2) rotated coherent state co-ordinates:

$$z'^{\alpha} = \left(\exp i(\theta^{a}\frac{\sigma^{a}}{2})\right)^{\alpha}{}_{\beta}\ z^{\beta}. \tag{A.13}$$

Therefore, under the SU(2)transformations the coherent states $|z\rangle \equiv |z_1,z_2\rangle$ transform amongst themselves on S^3 as the constraint (A.8) remains invariant under (A.13). Again we define the operator:

$$\mathcal{O}_{[j]} \equiv \int d\mu(z)\left(|z\rangle_{[j]}\ {}_{[j]}\langle z|\right) = \int\ d^2z^1 d^2z^2\,\delta(\ |z^1|^2+|z^2|^2-1\ )\ |z\rangle_{[j]}\ {}_{[j]}\langle z|. \tag{A.14}$$

The operator $\mathcal{O}_{[j]}$ is invariant under all SU(2) transformations of the coherent states $|z\rangle_{[j]}$. Therefore,

$$\left[Q^{a},\mathcal{O}_{[j]}\right] = 0, \qquad \forall a = 1,2,....,8 \tag{A.15}$$

The Schur's Lemma implies that $\mathcal{O}_{[j]}$ is proportional to identity operator.

It is illustrative to briefly mention the standard group theoretical coherent state construction procedure [66]. We characterize the SU(2) group elements U by the Euler angles, i.e, $U(\theta,\phi,\psi) \equiv exp{-i\phi J_3} exp{-i\theta J_2} exp{-i\psi J_3}$. The SU(2) coherent states are constructed as:

$$|\theta,\phi,\psi\rangle_j\ =\ U(\theta,\phi,\psi)\ |j,j\rangle\ ,=\ \sum_{m=-j}^{+j} C_m(\theta,\phi,\psi)\ |j,m\rangle\ ,$$

The coefficients $C_m(\theta, \phi, \psi)$ are given by,

$$C_m(\theta,\phi,\psi) = e^{-i(m\phi+j\psi)}\big[\frac{2j!}{(j+m)!(j-m)!}\big]^{\frac{1}{2}}\big[sin\frac{\theta}{2}\big]^{j-m}\big[cos\frac{\theta}{2}\big]^{j+m}.$$

It is clear that the corresponding construction is difficult for higher SU(N) group as we need to know all the SU(N) representations, Euler angles and the group elements to implement this procedure. On the other hand, the coherent states in (A.10) are straightforward generalization of the Heisenberg-Weyl coherent states in (A.2) and bypass all the problems mentioned above. In the next section we further extend this simple coherent state construction to SU(N) with arbitrary N. As we will see the only new input required for this purpose is the replacement of SU(2) Schwinger bosons by SU(N) irreducible Schwinger bosons [34, 35].

A.2 SU(3) Coherent States

In order to construct the coherent states for SU(3), we consider the SU(3) irreps constructed out of SU(3) irreducible Schwinger bosons defined and constructed in [34, 35]. Note that, the SU(3) irreducible Schwinger bosons used as prepotential operators in Chapter 5 of this thesis are not exactly the same which we are going to use here. Earlier we have used the 3 and 3^* irreps of SU(3) as the fundamental one and have constructed all the irreps out of it. Here instead we take two different 3 irrep as fundamental. As already mentioned while discussing the prepotential formulation for SU(N) lattice gauge theory both the schemes are exactly equivalent. As constructed in section 6.4.1, the irreducible Schwinger bosons $A^\dagger_\alpha[1], A^\dagger_\alpha[2]$ with $\alpha = 1, 2, 3$ for SU(3), constructed in (6.22), do commute with the constraint (6.20) and creates the SU(3) irreps as a monomial state given below:

$$\begin{aligned} \big|\alpha_1\alpha_2\ldots\alpha_{n_1};\beta_1\beta_2\ldots\beta_{n_2}\big\rangle_{[n_1 n_2]} \;\; \equiv \;\; & \Big(A^\dagger_{\beta_1}[2]A^\dagger_{\beta_2}[2]\ldots A^\dagger_{\beta_{n_2}}[2]\Big) \\ & \times\Big(A^\dagger_{\alpha_1}[1]A^\dagger_{\alpha_2}[1]\ldots A^\dagger_{\alpha_{n_1}}[1]\Big)\big|0\big\rangle. \qquad \text{(A.16)} \end{aligned}$$

The monomial states in (A.16) belong to $[n_1, n_2]$ irreducible representation of SU(3). This monomial state directly creates the SU(3) Young tableau with n_1 and n_2 boxes in the first and second rows respectively as shown in Figure A.2. . We now exploit this

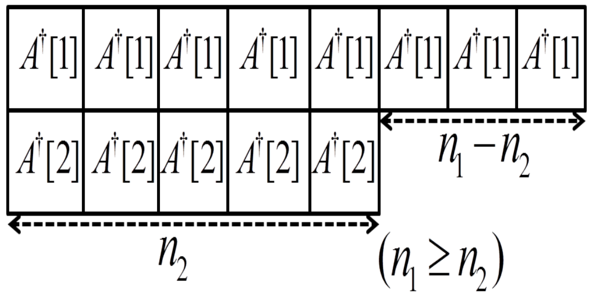

Figure A.2: SU(3) Young table in the $[n_1, n_2]$ representation. The monomial state (A.16) in terms of SU(3) irreducible Schwinger bosons carries all the symmetries of this SU(3) Young tableau.

simple construction of SU(3) irreps. to further extend the definition of Heisenberg Weyl, SU(2) coherent states (A.2) and (A.10) to SU(3) group.

A.2.1 Construction of SU(3) Coherent States

Similar to SU(2) case (A.8) and (A.9), the eight dimensional SU(3) group manifold can be characterized by two complex triplets: $z_\alpha[1]$ and $z_\alpha[2]$ $(\alpha = 1, 2, 3)$ which satisfy the orthonormality constraints:

$$\bar{z}[1] \cdot z[1] = 1 = \bar{z}[2] \cdot z[2] \quad , \quad \bar{z}[1] \cdot z[2] = 0. \tag{A.17}$$

This is because any SU(3) matrix $\mathcal{U}_3$ can be written as:

$$\mathcal{U}_3 = \begin{pmatrix} z_1[1] & z_1[2] & (\bar{z}[1] \wedge \bar{z}[2])_1 \\ z_2[1] & z_2[2] & (\bar{z}[1] \wedge \bar{z}[2])_2 \\ z_3[1] & z_3[2] & (\bar{z}[1] \wedge \bar{z}[2])_3 \end{pmatrix} \tag{A.18}$$

with $\mathcal{U}_3\mathcal{U}_3^\dagger = \mathcal{U}_3^\dagger\mathcal{U}_3 = 1$ and $det(\mathcal{U}_3) = |\mathcal{U}_3| = 1$ due to the orthonormality constraints (A.17).

We define the SU(3) coherent states generating function as:

$$\big|z[1], z[2]\big\rangle \equiv \exp\big(z[2] \cdot A^\dagger[2]\big) \exp\big(z[1] \cdot A^\dagger[1]\big) \big|0\big\rangle \tag{A.19}$$

Note that this construction is SU(3) extension of SU(2) coherent state generating function (A.10). We can project SU(3) coherent state in the representation $[n_1, n_2]$ by considering the corresponding term in the generating function (A.19):

$$\big|z[1], z[2]\big\rangle_{[n_1,n_2]} \equiv \frac{\big(z[2]\cdot A^{\dagger}[2]\big)^{n_2}}{n_2!}\frac{\big(z[1]\cdot A^{\dagger}[1]\big)^{n_1}}{n_1!}\big|0\big\rangle$$
$$= \sum_{\alpha_1..\alpha_{n_1}=1}^{3}\sum_{\beta_1..\beta_{n_2}=1}^{3} F^{\alpha_1..\alpha_{n_1};\beta_1..\beta_{n_2}}\big(z[1], z[2]\big)\ \underbrace{\big|\alpha_1\alpha_2...\alpha_{n_1};\beta_1\beta_2...\beta_{n_2}\big\rangle_{[n_1n_2]}}_{SU(3)\ irrep.\ [n_1,n_2]}. \quad (A.20)$$

In (A.20) the SU(3) coherent state structure functions,

$$F^{\alpha_1..\alpha_{n_1};\beta_1..\beta_{n_2}}\big(z[1], z[2]\big) = \frac{1}{n_1!n_2!}z[1]^{\alpha_1}z[1]^{\alpha_2}\dots z[1]^{\alpha_{n_1}}z[2]^{\beta_1}z[2]^{\beta_2}\dots z[2]^{\beta_{n_2}}. \quad (A.21)$$

are analytic functions of SU(3) group manifold co-ordinates. Like in SU(2) case, the resolution of identity property follows from the group transformation laws. Using the SU(3) transformations, we find that the SU(3) coherent states transform as: $|z[1], z[2]\rangle_{[n_1,n_2]} \rightarrow |z'[1], z'[2]\rangle_{[n_1,n_2]}$ where

$$z'^{\alpha}[1] = \left(\exp i(\theta^{a}\frac{\lambda^{a}}{2})\right)^{\alpha}{}_{\beta}\ z^{\beta}[1], \qquad z'^{\alpha}[2] = \left(\exp i(\theta^{a}\frac{\lambda^{a}}{2})\right)^{\alpha}{}_{\beta}\ z^{\beta}[2]. \quad (A.22)$$

Again like in SU(2) case, $z[1]$ & $z[2]$ transform like SU(3) triplets, the orthonormality conditions (A.17) remains invariant under the SU(3) transformations. In other words, the coherent state (A.20) defined at a point $(z[1], z[2])$ transform to the coherent state at $(z'[1], z'[2])$ on the SU(3) group manifold. Therefore, the operator $\mathcal{O}_{[n_1,n_2]}$:

$$\mathcal{O}_{[n_1,n_2]} \equiv \int d\mu(z)\left(\big|z[1], z[2]\big\rangle_{[n_1,n_2]}\ {}_{[n_1,n_2]}\big\langle z[1], z[2]\big|\right) \quad (A.23)$$

with SU(3) Haar measure

$$\int d\mu(z) \equiv \left(\int d^2z[1]d^2z[2]\right)\left(\prod_{\alpha,\beta=1}^{2}\delta(z[\alpha].z^{*}[\beta]-\delta_{\alpha,\beta})\right)$$

is invariant under all SU(3) transformations (A.22):

$$[Q^{a}, \mathcal{O}_{[n_1,n_2]}] = 0, \qquad \forall a = 1, 2, ..., 8 \quad (A.24)$$

where, Q^{a}'s are the SU(3) generators:

$$Q^{a} = a^{\dagger}[1]\,\frac{\lambda^{a}}{2}\,a[1] + a^{\dagger}[2]\,\frac{\lambda^{a}}{2}\,a[2], \quad a = 1, 2, \cdots, 8. \quad (A.25)$$

In (A.25) $\lambda^{\rm a}$'s are the Gell-Mann matrices. Therefore, by Schur's Lemma $\mathcal{O}_{[n_1,n_2]}$ is proportional to identity operator.

The SU(3) coherent states (A.20) and the structure functions (A.21) are straightforward generalization of the SU(2) coherent states (A.10) and the corresponding structure functions (A.12) respectively. The latter, in turn, are SU(2) generalization of the oldest Heisenberg-Weyl or harmonic oscillator coherent states (A.2) and the associated structure functions (A.3).

A.3 SU(N) coherent state

Just like SU(3), SU(N) irreducible Schwinger bosons defined in [35] are an useful tool to construct coherent states which are really essential to study the states in the SU(N) invariant Hilbert space locally at each site for SU(N) lattice gauge theory. In this section we use the same irreducible Schwinger bosons defined earlier in section 6.4.1 as SU(N) prepotential operators (6.27, 6.29) just omitting the link index. As in the case of SU(3) representation the SU(N) irreps are again monomial of irreducible Schwinger boson operators acting on vacuum as the irreducible Schwinger bosons carry all the symmetries of an arbitrary SU(N) Young tableaux. i.e,

$$\begin{aligned}
&\left|\alpha_1^{[1]},\alpha_2^{[1]},\ldots,\alpha_{n_1}^{[1]};\alpha_1^{[2]},\alpha_2^{[2]},\ldots,\alpha_{n_2}^{[2]};\cdots,\alpha_1^{[N-1]},\alpha_2^{[N-1]}\ldots,\alpha_{n_{N-1}}^{[N-1]}\right\rangle_{[n_1,n_2,\cdots n_{N-1}]} \\
\equiv\; & \underbrace{\left(A^{\dagger\alpha_1^{[N-1]}}[N-1]\cdots A^{\dagger\alpha_{n_{N-1}}^{[N-1]}}[N-1]\right)}_{n_{N-1}\;\; of A^\dagger[N-1]}\cdots \\
&\cdots\underbrace{\left(A^{\dagger\alpha_1^{[2]}}[2]\cdots A^{\dagger\alpha_{n_2}^{[2]}}[2]\right)}_{n_2\;\; of A^\dagger[2]}\underbrace{\left(A^{\dagger\alpha_1^{[1]}}[1]\cdots A^{\dagger\alpha_{n_1}^{[1]}}[1]\right)}_{n_1\;\; of A^\dagger[1]}|0\rangle.
\end{aligned} \tag{A.26}$$

A.3.1 Construction of SU(N) Coherent States

Like in SU(2) and SU(3) cases in (A.8) and (A.17) respectively, we characterize the SU(N) group manifold by $N-1$ number of complex N-plets: $\{z_\alpha[i]\}$, $i=1,2,\ldots,N-1$ and

$\alpha = 1, 2, \ldots N$ following orthonormality constraints:

$$\bar{z}[\alpha] \cdot z[\beta] = \delta_{\alpha,\beta}. \tag{A.27}$$

With the above parametrization any SU(N) matrix has the following form:

$$\mathcal{U}_N = \begin{pmatrix} z_1[1] & z_1[2] & \ldots & \ldots & z_1[N-1] & (\bar{z}[1] \wedge \bar{z}[2] \wedge \ldots \wedge \bar{z}[N-1])_1 \\ z_2[1] & z_2[2] & \ldots & \ldots & z_2[N-1] & (\bar{z}[1] \wedge \bar{z}[2] \wedge \ldots \wedge \bar{z}[N-1])_2 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ z_N[1] & z_N[2] & \ldots & \ldots & z_N[N-1] & (\bar{z}[1] \wedge \bar{z}[2] \wedge \ldots \wedge \bar{z}[N-1])_N \end{pmatrix} \tag{A.28}$$

At this stage we generalize (A.11) and (A.20) to define the SU(N) coherent state generating function as:

$$\begin{aligned} &\big|z[1], z[2], \ldots z[N-1]\big\rangle \\ &\equiv \exp\big(z[N-1] \cdot A^\dagger[N-1]\big) \ldots \exp\big(z[2] \cdot A^\dagger[2]\big) \exp\big(z[1] \cdot A^\dagger[1]\big) \big|0\big\rangle. \end{aligned} \tag{A.29}$$

Note that the coherent state generating function (A.29) contains all possible irreducible representations of SU(N). Further, the expressions for SU(N+1) and SU(N) coherent states differ only by the last exponential factor in (A.29). Therefore, the present SU(N) coherent state construction is iterative in nature. Now, projecting out a specific coherent state denoted by the set of particular values of the SU(N) Casimirs, i.e., $A^\dagger[i] \cdot A[i]$ having eigenvalue n_i with $i = 1, 2, \cdots (N-1)$ and $n_1 \geq n_2 \geq \ldots \geq n_{N-1}$ we get the SU(N) coherent state in the irreducible representation $[n_1, n_2, \cdots, n_{N-1}]$:

$$\begin{aligned} &|z[1], z[2], \ldots z[N-1]\rangle_{[n_1, n_2 \cdots n_{N-1}]} \\ &\equiv \frac{\big(z[N-1] \cdot A^\dagger[N-1]\big)^{n_{N-1}}}{n_{N-1}!} \ldots \frac{\big(z[2] \cdot A^\dagger[2]\big)^{n_2}}{n_2!} \frac{\big(z[1] \cdot A^\dagger[1]\big)^{n_1}}{n_1!} \big|0\big\rangle \\ &= \sum_{\alpha_1^{[1]}, .., \alpha_{n_1}^{[1]}=1}^{N} \sum_{\alpha_1^{[2]}, .., \alpha_{n_2}^{[2]}=1}^{N} \sum_{\alpha_1^{[N-1]}, .., \alpha_{n_{N-1}}^{[N-1]}=1}^{N} F^{\alpha_1^{[1]} .. \alpha_{n_1}^{[1]} \cdots \alpha_1^{[N-1]} .. \alpha_{n_{N-1}}^{[N-1]}} \big(z[1], z[2] \cdots z[N-1]\big) \\ &\qquad \underbrace{\left|\alpha_1^{[1]} .. \alpha_{n_1}^{[1]} \cdots \alpha_1^{[N-1]} .. \alpha_{n_{N-1}}^{[N-1]}\right\rangle}_{SU(N)\ irrep.\ state\ (A.26)}{}_{[n_1, n_1 . n_{N-1}]} \end{aligned} \tag{A.30}$$

In (A.30) the SU(N) coherent state structure functions are given by:

$$F^{\alpha^{[1]}_1..\alpha^{[1]}_{n_1}\cdots\alpha^{[N-1]}_1..\alpha^{[N-1]}_{n_{N-1}}} = \frac{1}{n_1!n_2!\dots n_{N-1}!} z[1]^{\alpha^{[1]}_1}..\ z[1]^{\alpha^{[1]}_{n_1}}\cdots z[N-1]^{\alpha^{[N-1]}_1}..\ \ z[N-1]^{\alpha^{[N-1]}_{n_{N-1}}}. \quad (A.31)$$

The states in (A.30) depend smoothly on the SU(N) group manifold coordinates. We now check the resolution of identity. Like in the previous SU(2) and SU(3) sections, under SU(N) transformations all the $(N-1)$ coherent state co-ordinates $z[i]$ transform as N-plets:

$$z_\alpha[i] \to z'_\alpha[i] = z_\beta[i]\left(\exp i\sum_{\mathrm{a}=1}^{N^2-1}\theta^{\mathrm{a}}\Lambda^{\mathrm{a}}\right)^{\beta}_{\ \ \alpha}. \quad (A.32)$$

We again define the operator $\mathcal{O}_{[n_1,\cdots n_{N-1}]}$ as:

$$\mathcal{O}_{[n_1,..,n_{N-1}]} \equiv \int d\mu(z)\ \Big(\big|z[1],..,z[N-1]\big\rangle_{[n_1,..,n_{N-1}]}\ \ {}_{[n_1,..,n_{N-1}]}\big\langle z[1],..,z[N-1]\big|\Big) \quad (A.33)$$

In (A.33) $\int d\mu(z)$ is the SU(N) invariant Haar measure:

$$\int d\mu(z) \equiv \left[\prod_{\alpha=1}^{N-1}\int d^2z[\alpha]\right]\prod_{\alpha,\beta}\delta\left(z[\alpha].z^*[\beta]-\delta_{\alpha,\beta}\right).$$

Under SU(N) transformations (A.32), $\mathcal{O}_{[N]}$ remains invariant. Therefore,

$$[Q^{\mathrm{a}},\mathcal{O}_{[n_1,n_2,\cdots,n_{N-1}]}] = 0, \qquad \forall \mathrm{a} = 1,2,....,N^2-1, \quad (A.34)$$

where, Q^{a}'s are SU(N) generators defined in terms of these Schwinger bosons as:

$$Q^{\mathrm{a}} = \sum_{i=1}^{N-1} a^\dagger[i]\,\frac{\Lambda^{\mathrm{a}}}{2}\,a[i], \qquad\qquad \mathrm{a} = 1,2,\cdots,(N^2-1). \quad (A.35)$$

Above Λ^{a}'s are the generalization of Gell-Mann matrices for SU(N).
The Schur's Lemma implies:

$$\mathcal{O}_{[n_1,n_2,\cdots,n_{N-1}]} = I_{[n_1,n_2,\cdots,n_{N-1}]}. \quad (A.36)$$

In (A.36) $I_{[n_1,n_2,\cdots,n_{N-1}]}$ is proportional to identity operator in the irreducible representation subspace. We again emphasize that the SU(N) coherent states in (A.30) are the most straightforward extension of the Heisenberg Weyl, SU(2) and SU(3) coherent states in (A.2), (A.11) and (A.20) respectively.

Appendix B

Calculation of Clebsch Gordon coefficients for $SU(N) \times SU(N)$

The coupled representations of two or more SU(N) are of particular interest in the context of gauge invariant states of gauge theories. As obvious tool in this treatment are the Clebsch Gordan coefficients which is well studied for SU(2) group. This enabled us to construct a complete orthonormal gauge invariant Hilbert space for SU(2) locally at each site of the lattice in chapter 3. The major obstacle to generalize it for higher rank groups are the lack of knowledge of the Clebsch Gordan series and in particular Clebsch Gordan coefficients. In this appendix we develop a new technique for computing Clebsch Gordan coefficients. We discuss the simplest and well known SU(2) case in this approach first and then generalize that to arbitrary SU(N).

We discuss the constructions for simple $SU(2)$ group (section B.1) first and then generalize these ideas and techniques to $SU(N)$ group with arbitrary N (section B.2).

B.1 Representations of $SU(2) \times SU(2)$ and invariants

Consider the generators of $SU(2) \times SU(2)$ Lie algebra as:

$$J_1^{\mathrm{a}} \equiv \frac{1}{2}\, a_\alpha^\dagger\, (\sigma^{\mathrm{a}})_{\alpha\beta}\, a_\beta, \qquad J_2^{\mathrm{a}} \equiv \frac{1}{2}\, b_\alpha^\dagger\, (\sigma^{\mathrm{a}})_{\alpha\beta}\, b_\beta, \tag{B.1}$$

where σ^{a} denote the Pauli matrices, $(a_\alpha, a_\alpha^\dagger)$ and $(b_\alpha, b_\alpha^\dagger)$ with $\alpha = 1, 2$ are the two Schwinger boson doublets. The SU(2) Casimirs are:

$$\vec{J}_1 \cdot \vec{J}_1 \equiv \frac{\hat{n}_a}{2}\left(\frac{N_a}{2} + 1\right), \qquad \vec{J}_2 \cdot \vec{J}_2 \equiv \frac{N_b}{2}\left(\frac{N_b}{2} + 1\right). \tag{B.2}$$

In (B.2), $N_a = \vec{a}^\dagger \cdot \vec{a} = (a_1^\dagger a_1 + a_2^\dagger a_2)$ and $N_b = \vec{b}^\dagger \cdot \vec{b} = (b_1^\dagger b_1 + b_2^\dagger b_2)$ are the number operators with eigenvalues $n_a = n_a^1 + n_a^2$ and $n_b = n_b^1 + n_b^2$ respectively. The decoupled angular momentum states are:

$$|j_1, m_1\rangle = \frac{\left(a_1^\dagger\right)^{j_1+m_1}\left(a_2^\dagger\right)^{j_1-m_1}}{\sqrt{(j_1+m_1)!(j_1-m_1)!}}|0\rangle, \qquad |j_2, m_2\rangle = \frac{\left(b_1^\dagger\right)^{j_2+m_2}\left(b_2^\dagger\right)^{j_2-m_2}}{\sqrt{(j_2+m_2)!(j_2-m_2)!}}|0\rangle. \quad \text{(B.3)}$$

The representations of $SU(2)$ can also be characterized by the eigenvalues of the total occupation number operator as,

$$|n_a^1, n_a^2\rangle = \frac{\left(a_1^\dagger\right)^{n_a^1}\left(a_2^\dagger\right)^{n_a^2}}{\sqrt{n_a^1! n_a^2!}}|0\rangle, \qquad |n_b^1, n_b^2\rangle = \frac{\left(b_1^\dagger\right)^{n_b^1}\left(b_2^\dagger\right)^{n_b^2}}{\sqrt{n_b^1! n_b^2!}}|0\rangle. \quad \text{(B.4)}$$

In (B.4) $n_a^1 = j_1 + m_1, n_a^2 = j_1 - m_1, n_b^1 = j_2 + m_2, n_b^2 = j_2 - m_2$. The direct product states $|n_a^1, n_a^2\rangle \otimes |n_b^1, n_b^2\rangle$ will often be denoted by $\left|\begin{array}{cc} n_a^1 & n_a^2 \\ n_b^1 & n_b^2 \end{array}\right\rangle$. The total angular momentum generators are:

$$J_T^{\mathrm{a}} = J_1^{\mathrm{a}} + J_2^{\mathrm{a}}. \quad \text{(B.5)}$$

The corresponding group will be denoted by $SU(2)_T$. We now construct all possible $SU(2)_T$ invariants out of the two Schwinger boson doublets in (B.1). The first set of invariant operators is:

$$k_+ \equiv a^\dagger \cdot \tilde{b}^\dagger, \quad k_- \equiv a \cdot \tilde{b}, \quad k_0 = \frac{1}{2}(N_a + N_b + 2). \quad \text{(B.6)}$$

In (B.6) the invariants $k_\pm$ are the antisymmetric combination of the two doublets: $a^\dagger \cdot \tilde{b}^\dagger \equiv \epsilon_{\alpha\beta} a_\alpha^\dagger b_\beta^\dagger = (a_1^\dagger b_2^\dagger - a_2^\dagger b_1^\dagger)$ and $a \cdot \tilde{b} \equiv \epsilon_{\alpha\beta} a_\alpha b_\beta = (a_1 b_2 - a_2 b_1)$. It is easy to check that k_+, k_- and k_0 commute with $SU(2)_T$ generators J^{a} in (B.5) and satisfy Sp(2,R) algebra:

$$[k_-, k_+] = 2k_0, \qquad [k_0, k_\pm] = \pm k_\pm. \quad \text{(B.7)}$$

The Sp(2,R) algebra and its representations are discussed in section 5.4. We use the same representation here also.

Similarly, another $SU(2)_T$ invariant algebra is obtained by defining [33]:

$$\kappa_+ \equiv a^\dagger \cdot b, \quad \kappa_- \equiv b^\dagger \cdot a, \quad \kappa_0 \equiv \frac{1}{2}\left(N_a - N_b\right). \quad \text{(B.8)}$$

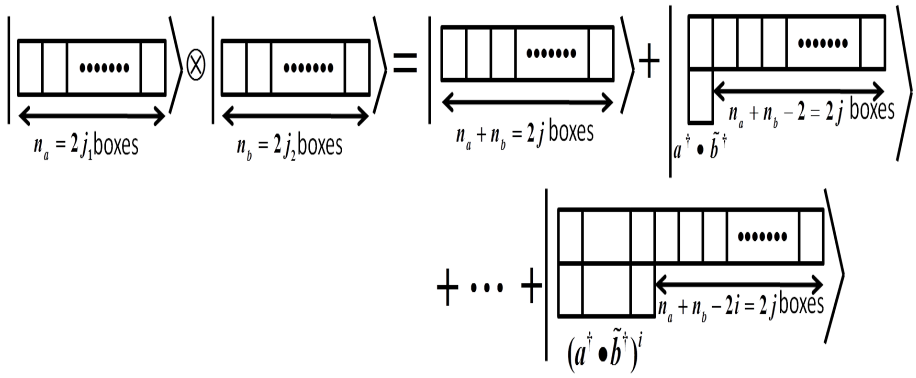

Figure B.1: Graphical or Young tableaue representation of the identity (B.11). The $SU(2) \otimes SU(2)$ coupled states on the right hand side can be directly obtained from the decoupled state by the corresponding projection operators (see (B.13)). The coupled states on the right hand sides also carry $Sp(2,R) \times SU(2)$ quantum numbers (see (B.25) and (B.26)).

These generators satisfy the standard $SU(2)$ algebra:

$$[\kappa_+, \kappa_-] = 2\kappa_0, \quad [\kappa_0, \kappa_\pm] = \pm\kappa_\pm. \tag{B.9}$$

It is easy to check that the $Sp(2,R)$ and $SU(2)$ generators in (B.6) and (B.8) respectively commute with each other as well as with $SU(2)_T$ in (B.5). Therefore, the coupled irreducible representations of $SU(2) \times SU(2)$ can also be labeled by the quantum numbers of the $Sp(2,R) \times SU(2)$ group (see (B.25) and (B.26)).

B.1.1 The Projection operators, invariants and symmetries of Irreps.

In this section we consider the coupled angular momentum states obtained by taking the direct product of two arbitrary angular momentum states $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ as shown in Figure B.1. The Young tableaue decomposition in Figure B.1 corresponds to the standard expansion of the decoupled basis in terms of the coupled basis:

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=j_1+j_2}^{|j_1-j_2|} C^{j,m}_{j_1,m_1;j_2,m_2} |j_1, j_2; j, m\rangle, \tag{B.10}$$

In (B.10) $m = m_1 + m_2$. The same series can also be obtained by defining projection operators $\mathcal{P}^j$ which directly project the decoupled state to a particular coupled state $|j_1, j_2; j, m\rangle$. In terms of projection operators, the expansion (B.10) takes the form:

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{j=j_1+j_2}^{|j_1-j_2|} \mathcal{P}^j |j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv \sum_{r=0}^{min(2j_1,2j_2)} \mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad \text{(B.11)}$$

In (B.11) r is the number of two boxes (invariants) on the right hand side of Figure B.1, i.e.,

$$r = j_1 + j_2 - j. \quad \text{(B.12)}$$

Comparing the series (B.11) with the standard expansion in (B.10) we get:

$$\mathcal{P}_r |j_1, m_1\rangle \otimes |j_2, m_2\rangle = C^{j,m}_{j_1,m_1;j_2,m_2} |j_1, j_2; j, m\rangle. \quad \text{(B.13)}$$

In (B.13) $j = j_1 + j_2 - r$ and $m = m_1 + m_2$. Taking the norms of each side of (B.13) and using $\mathcal{P}_r^2 = \mathcal{P}_r$ we get:

$$C^{j,m}_{j_1,m_1;j_2,m_2} = \sqrt{\langle j_1, j_2, m_1, m_2 | \mathcal{P}_r | j_1, j_2, m_1, m_2\rangle}. \quad \text{(B.14)}$$

In (B.14) we have used the notation $|j_1, j_2, m_1, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$. The Clebsch Gordan coefficients in (B.14) will be explicitly computed in section B.1.3.

We now construct the projection operators defined in (B.11). The Figure B.1 and (B.11) imply that the projection operators can only depend on the $SU(2)_T$ invariant operators $(N_a, N_b, k_\pm, \kappa_\pm)$ discussed in section 2. We first consider $r = 0$ $(j = j_1 + j_2)$ case. The Figure B.1 implies that $\mathcal{P}_0 (\equiv \mathcal{P})$ should completely symmetrize the $SU(2)$ indices so that $j = j_1 + j_2$. Therefore, we demand:

$$k_- \left(\mathcal{P} |j_1, m_1\rangle \otimes |j_2, m_2\rangle\right) = 0. \quad \text{(B.15)}$$

As $\mathcal{P}$ depends only on the $SU(2)_T$ invariant operators, we can write the most general form as:

$$\mathcal{P} = \sum_{q_1,q_2,q_3,q_4} l_{\{q\}}(N_a, N_b)(k_+)^{q_1}(k_-)^{q_2}(\kappa_+)^{q_3}(\kappa_-)^{q_4} \quad \text{(B.16)}$$

In (B.16) $l_{\{q\}} \equiv l_{q_1,q_2,q_3,q_4}(N_a, N_b)$ are the number operator dependent operators. Further, as the projection operator should not change the number of either a or b type oscillators, we get $q_1 = q_2,\ q_3 = q_4$. On the other hand, the identity $\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$ implies:

$$\kappa_+\kappa_- = N_aN_b - k_+k_-. \tag{B.17}$$

Thus all $SU(2)$ operators $\kappa_+\kappa_-$ in (B.16) can be removed in terms of Sp(2,R) operators k_+k_-. Therefore, the most general form of the projection operator is:

$$\mathcal{P} = \sum_{q=0}^{\infty} l_q(N_a, N_b)\ (k_+)^q(k_-)^q. \tag{B.18}$$

The constants l_q is computed using the constraint (B.15) and is obtained as:

$$l_q(N_a, N_b) = \frac{(-1)^q}{q!}\frac{(N_a + N_b - q)!}{(N_a + N_b)!} \tag{B.19}$$

Note that the constant term in (B.19) is chosen to be unity (i.e $l_0 = 1$) so that:

$$\mathcal{P}^2 = \mathcal{P}\mathcal{P} = \left(1 + \sum_{q=1}^{\infty} l_q(N_a, N_b)\ (k_+)^q(k_-)^q\right)\mathcal{P} = \mathcal{P}$$

as $k_-\mathcal{P} = 0$. The Figure B.1 now immediately implies that all other projection operators are of the form:

$$\mathcal{P}_r = N_r(k_+)^r\ \mathcal{P}\ (k_-)^r = N_r(k_+)^r\ \mathcal{P}\ (k_-)^r \tag{B.20}$$

The constant coefficients N_r are fixed by demanding that the operators $\mathcal{P}_r$ satisfy $\mathcal{P}_r^2 = \mathcal{P}_r$. We thus get:

$$N_r = \frac{(n_a + n_b - 2r + 1)!}{r!(n_a + n_b - r + 1)!} \tag{B.21}$$

Note that these coefficient can also be computed by demanding completeness property:

$$\sum_{r=0}^{\min(2j_1,2j_2)} \mathcal{P}_r = I. \tag{B.22}$$

The completeness property (B.22) is manifest in the defining expansion (B.11) itself. It is also easy to check that the Hilbert spaces projected by different projection operators in (B.20) are orthogonal:

$$\mathcal{P}_r\mathcal{P}_s = \delta_{rs}\mathcal{P}_r, \quad r, s = 0, 1, 2, \cdots\ \min(2j_1, 2j_2). \tag{B.23}$$

In (B.23) we have used the Sp(2,R) commutation relation (B.6) and the constraints $k_-\mathcal{P} = 0\ (r > s)$, $\mathcal{P}k_+ = 0 (r < s)$. We note that the coupled angular momentum states $\mathcal{P}k_-^r|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ in the expansion (B.11) also belong to the Sp(2,R) irreducible representations with lowest Sp(2,R) magnetic quantum number $q = q_0 = (j_1 + j_2 - r + 1)$ as:

$$\begin{aligned} k_0\, \mathcal{P}\ (k_-)^r\, |j_1, m_1\rangle \otimes |j_2, m_2\rangle &= q_0\ \mathcal{P}\, (k_-)^r\, |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \\ k^2\ \mathcal{P}\, (k_-)^r\, |j_1, m_1\rangle \otimes |j_2, m_2\rangle &= q_0(1 - q_0)\ \mathcal{P}\, (k_-)^r\, |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \end{aligned} \tag{B.24}$$

To get the second eigenvalue equation we have used $k_-\mathcal{P} = 0$ to replace $k^2 (\equiv \frac{1}{2}(k_+k_- + k_-k_+) - k_0^2)$ by $\frac{1}{2}[k_-, k_+] - k_0^2 = k_0(1 - k_0)$. The equations (B.24) immediately imply:

$$\begin{aligned} k_0\ \mathcal{P}_r\ |j_1, m_1\rangle \otimes |j_2, m_2\rangle &= (j_1 + j_2 + 1)\ \mathcal{P}_r|j_1, m_1\rangle \otimes |j_2, m_2\rangle. \\ k^2\ \mathcal{P}_r|j_1, m_1\rangle \otimes |j_2, m_2\rangle &= q_0(1 - q_0)\ \mathcal{P}_r|j_1, m_1\rangle \otimes |j_2, m_2\rangle. \end{aligned} \tag{B.25}$$

Similarly, it is easy to check that the quantum numbers of $SU(2)_T$ invariant $SU(2)$ group in (B.8) are:

$$\begin{aligned} \kappa_0\ \mathcal{P}_r\ |j_1, m_1\rangle \otimes |j_2, m_2\rangle &= (j_1 - j_2)\ \mathcal{P}_r|j_1, m_1\rangle \otimes |j_2, m_2\rangle, \\ \kappa^2\ \mathcal{P}_r|j_1, m_1\rangle \otimes |j_2, m_2\rangle &= j(j + 1)\ \mathcal{P}_r|j_1, m_1\rangle \otimes |j_2, m_2\rangle. \end{aligned} \tag{B.26}$$

Note that $j = j_1 + j_2 - r$ in (B.26).

B.1.2 $SU(2) \times SU(2)$ irreducible Schwinger bosons

It is known from [34,35] as well as from section 6.4.1 that, all possible $SU(N)$ irreducible representations can be written as monomials of $SU(N)$ irreducible Schwinger bosons [34, 35]. This irreducible Schwinger boson constructions are the $SU(N)$ extension of the Schwinger $SU(2)$ construction [33]. In this section we apply these ideas to construct the coupled states $|j_1, j_2, j, m\rangle$ in (B.13) as monomials of $SU(2)\times SU(2)$ irreducible Schwinger bosons (see equation (B.36)). The $SU(2) \times SU(2)$ irreducible Schwinger boson creation operators create states which satisfy $k_- = 0$ and therefore correspond to maximally symmetric states (or states with highest angular momentum). All other states can be constructed by applying the invariant operators on such maximally symmetric states. Note that this procedure is also illustrated by Figure B.1. The first coupled state on the right hand side with $n_a + n_b = 2j_1 + 2j_2$ is the maximally symmetric state. All other coupled states on the right hand side are obtained by multiplications of the invariant k_+ (i.e., two boxes arranged vertically in Figure B.1) on such maximally symmetric states. As in [34, 35], we define:

$$A^\dagger_\alpha \equiv a^\dagger_\alpha + f(N_a, N_b)k_+\tilde{b}_\alpha, \quad B^\dagger_\alpha \equiv b^\dagger_\alpha + g(N_a, N_b)k_+\tilde{a}_\alpha. \tag{B.27}$$

Note that by construction (B.27) the $SU(2) \times SU(2)$ transformation properties of $A^\dagger_\alpha$ and $B^\dagger_\alpha$ are exactly same as those of $a^\dagger_\alpha$ and $b^\dagger_\alpha$ respectively. We now demand:

$$k_-\ A^\dagger_\alpha\ \mathcal{P}|j_1, m_1\rangle \otimes |j_2, m_2\rangle = 0, \quad k_-\ B^\dagger_\alpha\ \mathcal{P}|j_1, m_1\rangle \otimes |j_2, m_2\rangle = 0. \tag{B.28}$$

The above constraints can be solved in terms of the unknown operator valued functions $f(N_a, N_b)$ and $g(N_a, N_b)$:

$$f(N_a, N_b) = -\frac{1}{(N_a + N_b)}, \qquad g(N_a, N_b) = \frac{1}{(N_a + N_b)}. \tag{B.29}$$

Note that the $f(N^a, N^b)$ and $g(N^a, N^b)$ in (B.29) are well defined as they always follow a creation operator in (B.27). As an example of the states created by the $SU(2)\times SU(2)$ irreducible Schwinger bosons we consider the state: $A^\dagger_\alpha B^\dagger_\beta|0\rangle = A^\dagger_\beta B^\dagger_\alpha|0\rangle = \frac{1}{2}\left(a^\dagger_\alpha b^\dagger_\beta + a^\dagger_\beta b^\dagger_\alpha\right)|0\rangle$. We note that it is already symmetric in the $SU(2)$ indices α and β and no explicit symmetrization is needed. Infact,

$$\begin{aligned} A^\dagger_1 B^\dagger_1|0\rangle &= \quad |j_1 = 1/2, j_2 = 1/2, j = 1, m = 1\rangle, \\ A^\dagger_1 B^\dagger_2|0\rangle &= \frac{1}{2}|j_1 = 1/2, j_2 = 1/2, j = 1, m = 0\rangle, \\ A^\dagger_2 B^\dagger_2|0\rangle &= \quad |j_1 = 1/2, j_2 = 1/2, j = 1, m = -1\rangle. \end{aligned} \tag{B.30}$$

The irreducible Schwinger bosons can also be directly constructed using the projection operators of the previous section as:

$$A^\dagger_\alpha \approx \mathcal{P}a^\dagger_\alpha, \qquad B^\dagger_\alpha \approx \mathcal{P}b^\dagger_\alpha. \tag{B.31}$$

In (B.31) $\approx$ implies weak equality. In other words the equations (B.31) are true only on the projected section of the Hilbert space which satisfies the constraint $k_- = 0$. The equivalence of (B.31) and (B.27) can be easily established by substituting $\mathcal{P}$ from (B.18) in (B.31) and noting that $l_1(N_a, N_b) = f(N_a, N_b) = -g(N_a, N_b)$. The completely symmetric states of $SU(2)_T$ can be easily defined through the irreducible Schwinger bosons:

$$|j_1, j_2, j = j_1 + j_2, m\rangle \equiv N^{j_1 m_1}_{j_2 m_2} \frac{\left(A^\dagger_1\right)^{j_1+m_1} \left(A^\dagger_2\right)^{j_1-m_1} \left(B^\dagger_1\right)^{j_2+m_2} \left(B^\dagger_2\right)^{j_2-m_2}}{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} |0\rangle. \quad (B.32)$$

To compute the normalization constant $N^{j_1,m_1}_{j_2,m_2}$ in (B.32) we note that the right hand side of the above equation can also be written in terms of decoupled states as:

$$\begin{aligned}
\left(N^{j_2 m_2}_{j_1 m_1}\right)^{-1} |j_1, j_2; j = j_1 + j_2, m\rangle &= \mathcal{P} \frac{\left(A^\dagger_1\right)^{j_1+m_1} \left(A^\dagger_2\right)^{j_1-m_1} \left(B^\dagger_1\right)^{j_2+m_2} \left(B^\dagger_2\right)^{j_2-m_2}}{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} |0\rangle \\
&= \mathcal{P} \frac{\left(a^\dagger_1\right)^{j_1+m_1} \left(a^\dagger_2\right)^{j_1-m_1} \left(b^\dagger_1\right)^{j_2+m_2} \left(b^\dagger_2\right)^{j_2-m_2}}{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} |0\rangle \\
&= \mathcal{P}|j_1, m_1\rangle \otimes |j_2, m_2\rangle. \qquad (B.33)
\end{aligned}$$

In the first step above we have introduced identity as $\mathcal{P}$. We then replace the irreducible Schwinger bosons by their defining equations (B.27) and used $\mathcal{P}k_+ = 0$ in the second step to get the decoupled states at the end. To compute the normalization $N^{j_1 m_1}_{j_2 m_2}$ in (B.32) we notice that the completely symmetric states are given by (B.13) at $r = 0$:

$$\mathcal{P}\ |j_1, m_1\rangle \otimes |j_2, m_2\rangle = C^{j=j_1+j_2,m}_{j_1,m_1,j_2,m_2} |j_1, j_2; j = j_1 + j_2, m\rangle.$$

Comparing this with (B.33) we get: $N^{j_1,m_1}_{j_2,m_2}\ C^{j=j_1+j_2,m}_{j_1,m_1,j_2,m_2} = 1$. Therefore,

$$N^{j_1 m_1}_{j_2 m_2} = \frac{1}{C^{j=j_1+j_2,m}_{j_1,m_1,j_2,m_2}} = \left[\frac{(2j_1+2j_2)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}{(2j_1)!(2j_2)!(j_1+j_2+m_1+m_2)!(j_1+j_2-m_1-m_2)!}\right]^{1/2} \quad (B.34)$$

For example, we put $j_1 = 2, m_1 = 0, j_2 = 1, m_2 = 0$ in (B.32) and replace the irreducible Schwinger bosons by their defining equations (B.27) and (B.29) to get:

$$\begin{aligned}
&|j_1 = 2, j_2 = 1, j = 3, m = 0\rangle \\
&= N^{20}_{10} \left[\frac{3}{5}|2,0\rangle|1,0\rangle + \frac{\sqrt{3}}{5}|2,-1\rangle|1,1\rangle + \frac{\sqrt{3}}{5}|2,1\rangle|1,-1\rangle\right]. \qquad (B.35)
\end{aligned}$$

Therefore, explicit normalization of the above state gives: $N_{10}^{20} = \sqrt{\frac{5}{3}}$ which is also the value obtained by (B.34) with $C_{j_1=2,m_1=0;j_2=1,m_2=0}^{\;j=3,m=0} = \sqrt{\frac{3}{5}}$. With this value of normalization, the expansion (B.35) further gives:

$$C_{j_1=2,m_1=-1;j_2=1,m_2=1}^{\;j=3,m=0} = C_{j_1=2,m_1=1;j_2=1,m_2=-1}^{\;j=3,m=0} = \sqrt{\frac{1}{5}}.$$

The same values are also obtained from the Clebsch Gordan series (B.47) obtained using the invariants in the next section. The above example provides a self consistency check on the procedure. The discussions in the previous section imply that an arbitrary coupled state can be written as:

$$|j_1, j_2; j, m\rangle = \mathcal{N}_{j_1,j_2}^{\;j}\,(k_+)^{j_1+j_2-j}\,|(j_1-j_2+j)/2, (j_2-j_1+j)/2; j, m\rangle. \qquad \text{(B.36)}$$

Note that the state $|(j_1-j_2+j)/2, (j_2-j_1+j)/2; j, m\rangle$ is maximally symmetric and is of the form (B.32). The normalization constants $\mathcal{N}_{j_1,j_2}^{\;j}$ can be easily computed using the commutation relations (B.7) as $k_-|(j_1-j_2+j)/2, (j_2-j_1+j)/2; j, m\rangle = 0$. They are given by:

$$\mathcal{N}_{j_1,j_2}^{\;j} = \sqrt{\frac{(2j_1+2j_2+1)!}{(j_1+j_2-j)!(3j_1+3j_2-j+1)!}}.$$

We again emphasize that all possible $SU(2)\times SU(2)$ coupled states in (B.36) are monomials of the irreducible Schwinger bosons. All the symmetries of the coupled Young tableaues on the right hand side of Figure B.1 are already present in (B.36) and there is no need for explicit symmetrization or anti-symmetrization by hand. Thus the the irreducible Schwinger bosons (B.36) can be thought of as the generalization of $SU(2)$ Schwinger bosons (B.3) which directly lead to coupled angular momentum states.

The $SU(2)\times SU(2)$ irreducible Schwinger bosons satisfy the following algebra:

$$\begin{aligned} &\left[A^\dagger_\alpha, A^\dagger_\beta\right] = 0, \quad \left[B^\dagger_\alpha, B^\dagger_\beta\right] = 0, \quad \left[A^\dagger_\alpha, B^\dagger_\beta\right] = 0. \\ &\left[A_\alpha, A^\dagger_\beta\right] = \delta_{\alpha\beta} - \frac{1}{N_a+N_b+1} b^\dagger_\alpha b_\beta + \frac{1}{(N_a+N_b)(N_a+N_b+1)} k_+ \tilde{b}_\beta a_\alpha \\ &\left[B_\alpha, B^\dagger_\beta\right] = \delta_{\alpha\beta} + \frac{1}{N_a+N_b+1} a^\dagger_\alpha a_\beta - \frac{1}{(N_a+N_b)(N_a+N_b+1)} k_+ \tilde{a}_\beta b_\alpha. \end{aligned} \qquad \text{(B.37)}$$

B.1.3 The projection operators and $SU(2)$ Clebsch Gordan Coefficients

The Clebsch Gordan coefficients are given by the defining equation (B.13):

$$C^{j=j_1+j_2-r,m}_{j_1,m_1;j_2,m_2} = \langle j_1, j_2; j = j_1 + j_2 - r, m|\mathcal{P}_r|j_1, m_1, j_2, m_2\rangle \tag{B.38}$$

This can be rewritten as:

$$\begin{aligned} C^{j=j_1+j_2-r,m}_{j_1,m_1;j_2,m_2} &= \frac{\langle j_1, m'_1 = j_1, j_2, m'_2 = m - j_1|\mathcal{P}_r\mathcal{P}_r|j_1, m_1, j_2, m_2\rangle}{C^{j=j_1+j_2-r,m}_{j_1,m'_1=j_1;j_2,m'_2=m-j_1}} \\ &= \frac{\langle j_1, j_1, j_2, m - j_1|\mathcal{P}_r|j_1, m_1, j_2, m_2\rangle}{[\langle j_1, j_1, j_2, m - j_1|\mathcal{P}_r|j_1, j_1, j_2, m - j_1\rangle]^{\frac{1}{2}}} \end{aligned} \tag{B.39}$$

We have used $\mathcal{P}_r^2 = \mathcal{P}_r$ in (B.39). The numerator in (B.39) is:

$$\begin{aligned} &\langle j_1, j_1, j_2, m - j_1|\mathcal{P}_r|j_1, m_1, j_2, m_2\rangle \\ &= N_r \left\langle \begin{array}{cc} 2j_1 & 0 \\ j_2 + m - j_1 & j_1 + j_2 - m \end{array} \right| k_+^r \mathcal{P} k_-^r \left| \begin{array}{cc} j_1 + m_1 & j_1 - m_1 \\ j_2 + m_2 & j_2 - m_2 \end{array} \right\rangle \\ &= N_r \sum_q l_q(2j_1 - r, 2j_2 - r) \underbrace{\left\langle \begin{array}{cc} 2j_1 & 0 \\ j_2 + m - j_1 & j_1 + j_2 - m \end{array} \right| k_+^{q+r}\ k_-^{q+r} \left| \begin{array}{cc} j_1 + m_1 & j_1 - m_1 \\ j_2 + m_2 & j_2 - m_2 \end{array} \right\rangle}_{\equiv K(j_1,m_1,j_2,m_2,q,r)} \\ &= N_r \sum_q\ l_q(2j_1 - r, 2j_2 - r)\ K(j_1, m_1, j_2, m_2, q, r). \end{aligned} \tag{B.40}$$

In the first step we have written the decoupled angular momentum states in terms of the occupation number basis. In the second step we have substituted the expression of $\mathcal{P}$ with $n_a = 2j_1 - r$, $n_b = 2j_2 - r$ for the coefficient l_q in (4.9). Note that the matrix elements K can be easily computed as both k_+^{q+r} and k_-^{q+r} in (B.40) can be replaced by monomials of harmonic oscillator creation and annihilation operators respectively:

$$k_+^{q+r} \to \left(a_1^\dagger\ b_2^\dagger\right)^{q+r}, \qquad k_-^{q+r} \to (-1)^{q+r-s}\ {}^{q+r}C_s\ (a_1 b_2)^s\, (a_2 b_1)^{q+r-s}.$$

Above $s = q + r + m_1 - j_1$. Substituting these monomials in (B.40) leads to:

$$K = \frac{(-1)^{q+r-s}(q + r)!}{s!(q + r - s)!(j_1 + m_1 - s)!(j_2 - m_2 - s)!(j_1 - m_1 - q - r + s)!(j_2 + m_2 - q - r + s)!}$$

$$\times\sqrt{(2j_1)!(j_2 + m - j_1)!(j_2 - m + j_1)!(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!}. \tag{B.41}$$

Substituting N_r, $l_q(2j_1-r, 2j_2-r)$ from (4.9) and K from above with $s = q+r+m_1-j_1$, the matrix element (B.40) takes the form:

$$\langle j_1, j_1, j_2, m-j_1|\mathcal{P}_r|j_1, m_1, j_2, m_2\rangle = \frac{(2j_1+2j_2-2r+1)!}{r!(2j_1+2j_2-r+1)!(2j_1+2j_2-2r)!}$$

$$\times\sqrt{\frac{(2j_1)!(j_2-m+j_1)!(j_1+m_1)!(j_2+m_2)!(j_2-m_2)!}{(j_2+m-j_1)!(j_1-m_1)!}}$$

$$\sum_{q=0}^{\min(2j_1-r, 2j_2-r)} \frac{(-1)^{q+j_1-m_1}(q+r)!(2j_1+2j_2-2r-q)!}{(q+r+m_1-j_1)!(j_1+j_2-m-q-r)!(2j_1-q-r)!} \quad \text{(B.42)}$$

Putting $r = j_1+j_2-j$ in the above equation we get:

$$\langle j_1, j_1, j_2, m-j_1|\mathcal{P}_j|j_1, m_1, j_2, m_2\rangle = \left[\frac{(2j+1)!}{(2j)!(j_1+j_2-j)!(j_1+j_2+j+1)!}\right.$$

$$\left.\sqrt{\frac{(2j_1)!(j_1+j_2-m)!(j_1+m_1)!(j_2+m_2)!(j_2-m_2)!}{(j_1-m_1)!(j_2-j_1+m)!}}\right]$$

$$\sum_{q=0}^{\min(j_1-j_2+j, j_2-j_1+j)} \frac{(-1)^{q+j_1-m_1}}{(q)!}\frac{(j_1+j_2-j+q)!(2j-q)!}{(j_2-j+q+m_1)!(j_1-j_2+j-q)!(j-m-q)!} \quad \text{(B.43)}$$

For the denominator of (B.39), we substitute $m_1 = j_1$ and $m_2 = m-j_1$ in (B.43) to obtain,

$$\langle j_1, j_1, j_2, m-j_1|\mathcal{P}_r|j_1, j_1, j_2, m-j_1\rangle$$

$$= \frac{(2j+1)!(2j_1)!(j_2+j_1-m)!}{(2j)!(j_1+j_2-j)!(j_1+j_2+j+1)!}\sum_{q=0}^{q_{\max}}\frac{(-1)^q}{q!}\frac{(2j-q)!}{(j_1-j_2+j-q)!(j-m-q)!}. \quad \text{(B.44)}$$

In (B.44) the upper limit on the sum over q is $q_{\max} \equiv min(2j_1-r, 2j_2-r) = min(j_1-j_2+j, j_2-j_1+j)$. This above series in q is summed using the formula:

$$\sum_{q=0}\frac{(-1)^q}{q!}\frac{(C-q)!}{(A-q)!(B-q)!} = \frac{C!}{A!B!}\times\frac{(C-A)!(C-B)!}{C!(C-A-B)!} \quad \text{(B.45)}$$

Finally, the denominator in (B.39) is:

$$\sqrt{\langle j_1, j_1, j_2, m-j_1|\mathcal{P}_r|j_1, j_1, j_2, m-j_1\rangle}$$

$$= \sqrt{\frac{(j+m)!(j_2-j_1+j)!(2j+1)!(2j_1)!(j_2+j_1-m)!}{(2j)!(j_1+j_2-j)!(j_1+j_2+j+1)!(j-m)!(j_2-j_1+m)!(j_1-j_2+j)!}} \quad \text{(B.46)}$$

The final expression of the Clebsch Gordon coefficient in (B.47) is now obtained by dividing (B.43) by (B.46) as:

$$C^{\;j,m}_{j_1,m_1;j_2,m_2} = \delta_{m,m_1+m_2}\sqrt{\frac{(j_1-j_2+j)!(j_2-m_2)!(j_2+m_2)!(j_1+m_1)!(2j+1)(j-m)!}{(j_1+j_2+j+1)!(j_2-j_1+j)!(j_1+j_2-j)!(j_1-m_1)!(j+m)!}}$$
$$\sum_{q=0}^{\min\{j_1-j_2+j,j_2-j_1+j\}} \frac{(-1)^{q+j_1-m_1}(2j-q)!(j_1+j_2-j+q)!}{q!(j_1-j_2+j-q)!(j-m-q)!(j_2-j+m_1+q)!} \quad \text{(B.47)}$$

The series representing Clebsch Gordon coefficient in (B.47) matches with the expansion given in [38]. In section B.2.2 this $SU(2)$ computation will be extended to $SU(N)$.

B.2 Invariants and representations of $SU(N) \times SU(N)$

We now generalize the previous $SU(2)$ ideas and techniques to direct product of two conjugate representations of $SU(N)$. For simplicity we choose these to be N and N^* representations of $SU(N)$. We write the corresponding generators as:

$$Q_1^{\rm a} \equiv \frac{1}{2}\, a^{\dagger\alpha}\, (\Lambda^{\rm a})_\alpha^{\;\beta}\, a_\beta, \qquad Q_2^{\rm a} \equiv -\,\frac{1}{2}\, b^\dagger_\alpha\, (\tilde{\Lambda}^{\rm a})^\alpha_{\;\beta}\, b^\beta. \quad \text{(B.48)}$$

In (B.48) $\mathrm{a} = 1,2,\cdots(N^2-1)$ and $\alpha,\beta = 1,2\cdots N$. $\Lambda^{\rm a}$'s are the generalized Gell-Mann matrices for N-plets of $SU(N)$ and $-\tilde{\Lambda}^{\rm a}$ are the dual matrices corresponding to the N^*-plets of $SU(N)$. From (B.48) it is clear that $a^\dagger$'s transform as N under one $SU(N)$ and $b^\dagger$'s transform as N^* under another $SU(N)$[1]. Like in $SU(2)$ case (B.4) the decoupled N and N^* irreducible representations are:

$$|\{n_a\}\rangle \equiv |n_a^1, n_a^2, \cdots n_a^N\rangle = \frac{\left(a^{\dagger 1}\right)^{n_a^1}\left(a^{\dagger 2}\right)^{n_a^2}\cdots\left(a^{\dagger N}\right)^{n_a^N}}{\sqrt{n_a^1! n_a^2! \cdots n_a^N!}}|0\rangle,$$
$$|\{n_b\}\rangle \equiv |n_b^1, n_b^2, \cdots n_b^N\rangle = \frac{\left(b_1^\dagger\right)^{n_b^1}\left(b_2^\dagger\right)^{n_b^2}\cdots\left(b_N^\dagger\right)^{n_b^N}}{\sqrt{n_b^1! n_b^2! \cdots n_b^N!}}|0\rangle \quad \text{(B.49)}$$

In (B.49) $\{n\}$ represents N partitions of n. The two Casimirs are the two total number

[1]For $N \geq 3$ the N and N^* representations are not equivalent. Therefore, we now use upper $a^{\dagger\alpha}$ and lower $b^\dagger_\alpha$ indices to differentiate between the two conjugate representations.

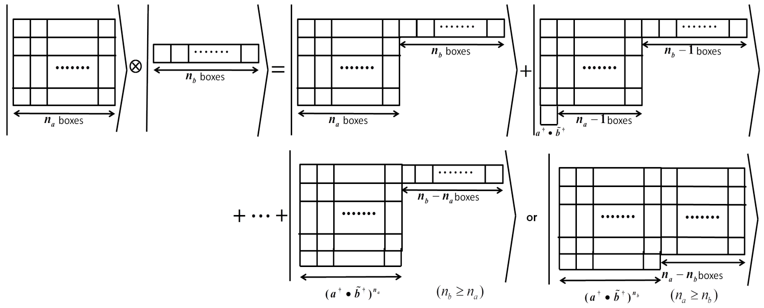

Figure B.2: Graphical or Young tableau representation of the identity (B.55). The coupled $SU(N) \times SU(N)$ states on the right hand side can be directly obtained by projection operators (see (B.56)) and also carry Sp(2,R) quantum numbers (see (B.63)).

operators N_a and N_b with eigenvalues n_a and n_b respectively:

$$n_a^1 + n_a^2 + \cdots + n_a^N = n_a, \qquad n_b^1 + n_b^2 + \cdots + n_b^N = n_b. \tag{B.50}$$

We will often denote the $SU(N)$ direct product state $|\{n_a\}\rangle \otimes |\{n_b\}\rangle$ by $\left| \begin{matrix} n_a^1 & n_a^2 & \cdots & n_a^N \\ n_b^1 & n_b^2 & \cdots & n_b^N \end{matrix} \right\rangle$. As in the case of $SU(2)$ (B.5), we define the total $SU(N)$ flux operators:

$$Q_T^a = Q_1^a + Q_2^a. \tag{B.51}$$

The corresponding group will be denoted by $SU(N)_T$. At this stage we can also define the coupled $SU(N) \times SU(N)$ states through the Clebsch Gordan decomposition (see Figure B.2) as:

$$|\{n_a\}\rangle \otimes |\{n_b\}\rangle = \sum_{r=0}^{min(n_a,n_b)} C^{\;r}_{\{n_a\},\{n_b\}} \; |\{n_a - r\}\{n_b - r\}; r\rangle \tag{B.52}$$

As in Figure B.2, the coupled states denoted by $|\{n_a - r\}\{n_b - r\}; r\rangle$ in (B.52) represents the invariant operator $(k_+)^r$ acting on the completely traceless tensor state of rank $(n_a - r, n_b - r)$.

We now define the following $SU(N) \times SU(N)$ invariant operators:

$$k_+ \equiv a^\dagger \cdot b^\dagger, \quad k_- \equiv a \cdot b, \quad k_0 = \frac{1}{2}(N_a + N_b + N). \tag{B.53}$$

In(B.53) the invariants are the scalar products of N and N^* representations: $a^\dagger \cdot b^\dagger = (a^{\dagger 1} b_1^\dagger + a^{\dagger 2} b_2^\dagger + \cdots + a^{\dagger N} b_N^\dagger)$ and $a \cdot b = (a_1 b^1 + a_2 b^2 + \cdots + a_N b^N)$. It is easy to check that they again satisfy Sp(2,R) algebra (B.7):

$$[k_-, k_+] = 2k_0, \qquad [k_0, k_\pm] = \pm k_\pm. \tag{B.54}$$

Like in $SU(2)$ case, the $SU(N) \times SU(N)$ projection operators are defined as:

$$|\{n_a\}\rangle \otimes |\{n_b\}\rangle = \sum_{r=0}^{min(n_a,n_b)} \mathcal{P}_r \; |\{n_a\}\rangle \otimes |\{n_b\}\rangle. \tag{B.55}$$

Comparing (B.52) with (B.55) we get:

$$\mathcal{P}_r|\{n_a\}\rangle \otimes |\{n_b\}\rangle \equiv C^{\;\;r}_{\{n_a\},\{n_b\}}|\{n_a - r\},\{n_b - r\}; r\rangle. \tag{B.56}$$

In (B.56) $C^{\;\;r}_{\{n_a\},\{n_b\}}$ are the Clebsch Gordan coefficients. Taking the norms on both the sides of (B.56) we get a simple expression for the $SU(N)$ Clebsch Gordan coefficients:

$$C^{\;\;r}_{\{n_a\},\{n_b\}} = \sqrt{\langle\{n_a\}| \otimes \langle\{n_b\}| \; \mathcal{P}_r \; |\{n_a\}\rangle \otimes |\{n_b\}\rangle} \tag{B.57}$$

We will explicitly compute these coefficients in section B.2.2.

We now construct the projection operators defined in (B.56). Note that the condition of tracelesness is exactly same as demanding the constraint $k_- = 0$. Using this fact and the invariant algebra (B.54) the $SU(N)$ projection operator $\mathcal{P}_0$ can be easily constructed like in $SU(2)$ case as:

$$\mathcal{P} \equiv \mathcal{P}_0 = \sum_{q=0}^{\infty} L_q\left(N^a, N^b\right)(k_+)^q(k_-)^q \tag{B.58}$$

where

$$L_q\left(N_a, N_b\right) = \frac{(-1)^q}{q!}\frac{(N_a + N_b + N - 2 - q)!}{(N_a + N_b + N - 2)!}.) \tag{B.59}$$

Again, the projection operator in (B.58) satisfies

$$\mathcal{P}^2 = \mathcal{P}\mathcal{P} = \left(1 - \frac{1}{N_a + N_b + N - 2}k_+k_- + \cdots\right)\mathcal{P} = \mathcal{P},$$

as $k_-\mathcal{P} = 0$: Note that the $SU(N)$ projection operator (B.58) reduces to the $SU(2)$ projection operator (B.18) at $N = 2$. Like in $SU(2)$ case, all other projection operators in (B.55) or equivalently in Figure B.2 are of the form:

$$\mathcal{P}_r = N_r \;\; (k_+)^r \; \mathcal{P}_0 \; (k_-)^r = N_r \;\; (k_+)^r \; \mathcal{P} \; (k_-)^r \tag{B.60}$$

The constant coefficients N_r are fixed by demanding that the operators $\mathcal{P}_r$ satisfy: $\mathcal{P}_r^2 = \mathcal{P}_r$ and are obtained as:

$$N_r = \frac{(n_a + n_b + N - 2r - 1)!}{r!(n_a + n_b + N - r - 1)!} \tag{B.61}$$

As expected, (B.61) reduces to (B.21) at $N = 2$. Like in $SU(2)$ case the projection operators satisfy the orthogonality and completeness properties:

$$\sum_{r=0}^{\min(n_a,n_b)} \mathcal{P}_r = I, \quad \mathcal{P}_r\mathcal{P}_s = \delta_{rs}\mathcal{P}_r, \quad r, s = 0, 1, 2, \cdots \min(n_a, n_b). \tag{B.62}$$

The orthogonality relation can be proven exactly like in the $SU(2)$ case (see (B.23)) and the completeness relation is also manifest in (B.55).

Note that the coupled states are also eigenstates of k_0 and k^2 and carry the following Sp(2,R) quantum numbers:

$$\begin{aligned} k_0\ \mathcal{P}_r\ |\{n_a\}\rangle \otimes |\{n_b\}\rangle &= \frac{1}{2}(n_a + n_b + N)\ \mathcal{P}_r|\{n_a\}\rangle \otimes |\{n_b\}\rangle. \\ k^2\ \mathcal{P}_r|\{n_a\}\rangle \otimes |\{n_b\}\rangle &= q_0(1 - q_0)\ \mathcal{P}_r|\{n_a\}\rangle \otimes |\{n_b\}\rangle, \end{aligned} \tag{B.63}$$

where, $q_0 \equiv \frac{1}{2}(n_a + n_b + N - 2r)$. It reduces to its $SU(2)$ value for $N = 2, n_a = 2j_1, n_b = 2j_2$.

B.2.1 $SU(N) \times SU(N)$ irreducible Schwinger bosons

Like in the section B.1.2 (also see [34, 35]), we define:

$$A^{\dagger\alpha} \equiv a^{\dagger\alpha} + F(N_a, N_b)k_+b^\alpha, \quad B^\dagger_\alpha \equiv b^\dagger_\alpha + G(N_a, N_b)k_+a_\alpha. \tag{B.64}$$

Note that by construction (B.64) the $SU(N)_T$ transformation properties of $A^{\dagger\alpha}$ and $B^\dagger_\alpha$ are exactly same as those of $a^{\dagger\alpha}$ and $b^\dagger_\alpha$ respectively. We now demand:

$$k_-\ A^{\dagger\alpha}\ \mathcal{P}|\{n^i_a\}\rangle \otimes |\{n^i_b\}\rangle = 0, \quad k_-\ B^\dagger_\alpha\ \mathcal{P}|\{n^i_a\}\rangle \otimes |\{n^i_b\}\rangle = 0. \tag{B.65}$$

The above constraints can be solved in terms of the unknown functions $F(N_a, N_b)$ and $G(N_a, N_b)$:

$$F(N_a, N_b) = G(N_a, N_b) = -\frac{1}{(N_a + N_b + N - 2)}. \tag{B.66}$$

The first state of the Clebsch Gordon series for $SU(N) \times SU(N)$ as given in Figure B.2 can be easily defined through the irreducible Schwinger bosons:

$$\Big|\{n_a\},\{n_b\};r=0\Big\rangle \equiv N^{\{n_a\}}_{\{n_b\}}\frac{\left(A^{\dagger 1}\right)^{n_a^1}\cdots\left(A^{\dagger N}\right)^{n_a^N}\left(B_1^\dagger\right)^{n_b^1}\cdots\left(B_N^\dagger\right)^{n_b^N}}{\sqrt{(n_a^1)!\,(n_a^2)!\cdots(n_a^N)!\,(n_b^1)!\,(n_b^2)!\cdots(n_b^N)!}}\,|0\rangle$$

$$= N^{\{n_a\}}_{\{n_b\}}\mathcal{P}\frac{\left(a^{\dagger 1}\right)^{n_a^1}\cdots\left(a^{\dagger N}\right)^{n_a^N}\left(b_1^\dagger\right)^{n_b^1}\cdots\left(b_N^\dagger\right)^{n_b^N}}{\sqrt{(n_a^1)!\,(n_a^2)!\cdots(n_a^N)!\,(n_b^1)!\,(n_b^2)!\cdots(n_b^N)!}}|0\rangle$$

$$= N^{\{n_a\}}_{\{n_b\}}\mathcal{P}\Big|\{n_a^i\}\Big\rangle\otimes\Big|\{n_b^i\}\Big\rangle \qquad \text{(B.67)}$$

The $r = 0$ state in (B.67) is the first coupled representation on the right hand side of Figure B.2. The $\mathcal{N}^{\{n_a\}}_{\{n_b\}}$ are the normalization constants. Again the construction (B.67) is the simplest and direct generalization of Schwinger boson construction (B.3) and (B.49) to $SU(N) \times SU(N)$ group. As an example we consider:

$$\left(N^{\{n_a=1\}}_{\{n_b=1\}}\right)^{-1}|\{n_a=1\},\{n_b=1\};r=0\rangle = A^{\dagger\alpha}B^\dagger_\beta|0\rangle = \left(a^{\dagger\alpha}b^\dagger_\beta - \frac{1}{N}\delta^\alpha_\beta k_+\right)|0\rangle.$$

Thus the tracelesness or equivalently the symmetries of Young tableaues are manifestly present in the definition of $SU(N) \times SU(N)$ irreducible Schwinger bosons.

Comparing (B.56) at $r = 0$ with (B.67) we get:

$$N^{\{n_a\}}_{\{n_b\}}\;C^{\;r=0}_{\{n_a\},\{n_b\}} = 1. \qquad \text{(B.68)}$$

Hence, like in $SU(2)$ case (B.34) the normalization factor $N^{\{n_a\}}_{\{n_b\}}$ is just the inverse of the CG coefficient at $r = 0$. This normalization can be calculated using (B.78) from section B.2.2. As an example we consider SU(3) states (B.49) with partitions: $n_a^1 = 1, n_a^3 = 0; n_b^1 = 1, n_b^2 = 1$, $n_b^3 = 0$. In (B.67) we replace the irreducible Schwinger bosons by their defining equation (B.64) and (B.29) to get,

$$\Big|\{n_a^1=1,n_a^2=0,n_a^3=0\},\{n_b^1=1,n_b^2=1,n_b^3=0\};r=0\Big\rangle = N^{1,0,0}_{1,1,0}\;\; A_1^\dagger B^{\dagger 1}B^{\dagger 2}\Big|0\Big\rangle$$

$$= N^{1,0,0}_{1,1,0}\;\left[\frac{3}{4}\left|\begin{matrix}1&0&0\\1&1&0\end{matrix}\right\rangle - \frac{\sqrt{2}}{4}\left|\begin{matrix}0&1&0\\0&2&0\end{matrix}\right\rangle - \frac{1}{4}\left|\begin{matrix}0&0&1\\0&1&1\end{matrix}\right\rangle\right]. \qquad \text{(B.69)}$$

Therefore, explicit normalization of the above state gives: $N^{1,0,0}_{1,1,0} = \sqrt{\frac{4}{3}}$. On the other hand, this normalization can also be computed by using (B.68) and the SU(3) Clebsch

Gordan expression (B.78) obtained in the next section. Putting the above values of occupation numbers and $r = 0$ in (B.78) we get:

$$\left(N^{1,0,0}_{1,1,0}\right)^{-1} = C^{r=0}_{\{n_a^1=1\},\{n_b^1=1,n_b^2=1\}} = \sqrt{\frac{2!}{4!3!3!}(4!-3!)} = \sqrt{\frac{3}{4}}.$$

Infact, at this stage we can cross check the other values of the $SU(N)$ Clebsch Gordan coefficients present in (B.69) with their values computed from the $SU(N)$ Clebsch Gordan expression (B.78) in the next section. The decomposition (B.69) implies

$$C^{\;\;r=0}_{\{010\}\{020\}} = -\sqrt{\frac{1}{6}} \quad and \quad C^{\;\;r=0}_{\{001\}\{011\}} = -\sqrt{\frac{1}{12}}.$$

As can be checked, these are also the values obtained from (B.78) after putting $N = 3$, various occupation numbers and $r = 0$. Thus the above simple state provides three self consistency checks on our procedure.

The discussions in the previous section and Figure B.2 imply that an arbitrary coupled state can be written as:

$$|\{n_a\},\{n_b\};r\rangle = \mathcal{N}^{\;\;r}_{n_a,n_b}\,(k_+)^r\,|\{n_a-r\},\{n_b-r\};r=0\rangle \tag{B.70}$$

The normalization constants $\mathcal{N}^{\;\;r}_{n_a,n_b}$ can be easily computed as $k_-|\{(n_a)\},\{(n_b)\};r=0\rangle = 0$ and are given by:

$$\mathcal{N}^{\;\;r}_{n_a,n_b} = \sqrt{\frac{(n_a+n_b+N-1)!}{r!(n_a+n_b+N+r-1)!}}.$$

We again emphasize that except the invariant term all the $SU(N)\times SU(N)$ coupled states in (B.36) are monomials of the irreducible Schwinger bosons. The present construction of coupled states is a straightforward generalization of the original construction to the decoupled $SU(2)$ angular momentum states (B.3).

B.2.2 The Projection operators and $SU(N)$ Clebsch Gordon Coefficients

We write (B.55) and (B.56) as

$$\big|\{n_a\}\big\rangle \otimes \big|\{n_b\}\big\rangle = \sum_{r=0}^{n} \mathcal{P}_r \big|\{n_a\}\big\rangle \otimes \big|\{n_b\}\big\rangle \equiv \sum_{r=0}^{n} C^{\;\;\;r}_{\{n_a\},\{n_b\}} \big|\{n_a-r\},\{n_b-r\};r\big\rangle \tag{B.71}$$

where, $n = \min(n_a, n_b)$. Hence the Clebsch Gordon Coefficients can be computed as in the $SU(2)$ case:

$$C^{\ r}_{\{n_a^i\},\{n_b^i\}} = \frac{\left\langle n_a, 0, \cdots 0\right| \otimes \left\langle \bar{n}_b^1, \bar{n}_b^2, \cdots, \bar{n}_b^N\right| \mathcal{P}_r \left|n_a^1, n_a^2, \cdots n_a^N\right\rangle \otimes \left|n_b^1, n_b^2, \cdots n_b^N\right\rangle}{\left[\left\langle n_a, 0, \cdots, 0\right| \otimes \left\langle n_b^1, \bar{n}_b^2, \cdots, \bar{n}_b^N\right| \mathcal{P}_r \left|n_a, 0, \cdots, 0\right\rangle \otimes \left|\bar{n}_b^1, \bar{n}_b^2, \cdots, \bar{n}_b^N\right\rangle\right]^{\frac{1}{2}}} \quad \text{(B.72)}$$

In the above equation $\{\bar{n}_b^1, \cdots \bar{n}_b^N\}$ are the values of the occupation numbers corresponding to the special choice $\{n_a^1 = n^a, 0, 0, \cdots, 0\}$ so that the total magnetic quantum numbers on both sides of the projection operator remain unchanged[2]: They are given by:

$$\bar{n}_b^1 = n_a - n_a^1 + n_b^1 \qquad and \qquad \bar{n}_b^i = n_b^i - n_a^i \qquad , i = 2, 3, ..., N$$

Similarly the matrix element in the numerator of $SU(N)$ Clebsch Gordon coefficient expression (B.72) is:

$$\begin{aligned} &\left\langle \{n_a^1 = n_a\}, \{\bar{n}_b\} \middle| \mathcal{P}_r \mathcal{P}_r \middle| \{n_a\}, \{n_b\} \right\rangle \\ = \; & N_r \sum_q l_q(n_a - r, n_b - r) \underbrace{\left\langle \begin{matrix} n_a & 0 & \dots & 0 \\ \bar{n}_b^1 & \bar{n}_b^2 & \dots & \bar{n}_b^N \end{matrix} \middle| k_+^{q+r} k_-^{q+r} \middle| \begin{matrix} n_a^1 & n_a^2 & \dots & n_a^N \\ n_b^1 & n_b^2 & \dots & n_b^N \end{matrix} \right\rangle}_{K(\{n_a\},\{n_b\},q,r)} \\ = \; & N_r \sum_q l_q(n_a - r, n_b - r) K\left(\{n_a\}, \{n_b\}, q, r\right) \end{aligned} \quad \text{(B.73)}$$

The matrix element $K(\{n_a\}, \{n_b\}, q, r)$ are calculated in the same way as in the $SU(2)$ case. In the computation of $K(\{n_a\}, \{n_b\}, q, r)$ in (B.73) k_+^{q+r} and k_-^{q+r} can be replaced by the following monomials of Schwinger bosons:

$$k_+^{q+r} \rightarrow (a_1^\dagger b^{\dagger 1})^{q+r}, \qquad k_-^{q+r} \rightarrow \frac{(q+r)!}{\beta_1! \dots \beta_N!} (a^1 b_1)^{\beta_1} (a^2 b_2)^{\beta_2} \dots (a^N b_N)^{\beta_N} \quad \text{(B.74)}$$

---

[2]Note that the $SU(N)$ states $|n^1, n^2, \cdots, n^N\rangle$ in (B.49) can also be characterized by $SU(N)$ Casimir $n = n^1 + n^2 + \cdots + n^N$ along with the "$SU(N)$ magnetic quantum numbers" $\{h^i\}(i = 1, 2 \cdots (N-1))$ as:

$$\begin{aligned} h_a^1 &= n_a^1 - n_a^2, & h_b^1 &= n_b^2 - n_b^1 \\ h_a^2 &= n_a^1 + n_a^2 - 2n_a^3, & h_b^2 &= 2n_b^3 - n_b^1 - n_b^2 \\ &\vdots \\ h_a^{N-1} &= n_a^1 + n_a^2 + \ldots + n_a^{N-1} - (N-1)n_a^N, & h_b^{N-1} &= (N-1)n_b^N - n_b^1 - n_b^2 - \ldots - n_b^{N-1}. \end{aligned}$$

Equating the occupation numbers in the matrix element in (B.73) we get:

$$\beta_1 = q + r + n_a^1 - n_a, \quad \beta_2 = n_a^2, \quad \beta_3 = n_a^3, \qquad \ldots, \beta_N = n_a^N,$$

leading to:

$$\begin{aligned} K\left(\{n_a\}, \{n_b\}, q, r\right) &= \sqrt{n_a! \bar{n}_b^1! \bar{n}_b^2! \ldots \bar{n}_b^N! n_a^1! n_a^2! \ldots n_a^N! n_b^1! n_b^2! \ldots n_b^N!} \\ &\times \frac{(q+r)!}{(n_a^1 - n_a + q + r)! n_a^2! \ldots n_a^N!} \\ &\times \frac{1}{(n_a - q - r)!(\bar{n}_b^1 - q - r)! \bar{n}_b^2! \ldots \bar{n}_b^N!} \end{aligned} \tag{B.75}$$

Now substituting the values of N_r and $l_q(n_a - r, n_b - r)$ obtained as:

$$\begin{aligned} N_r &= \frac{(n_a + n_b + N - 2r - 1)!}{r!(n_a + n_b + N - r - 1)!} \\ l_q(n_a - r, n_b - r) &= \frac{(-1)^q}{q!} \frac{(n_a + n_b - q - 2r)!}{(n_a + n_b - 2r)!} \end{aligned}$$

and the matrix element K from above we finally get the numerator of (B.72) as:

$$\begin{aligned} \Big\langle \{n_a^1 = n_a\}, \{\bar{n}_b\} \Big| \mathcal{P}_r \Big| \{n_a\}, \{n_b\} \Big\rangle &= \frac{(n_a + n_b + N - 2r - 1)}{r!(n_a + n_b + N - r - 1)!} \sqrt{\frac{n_a! \bar{n}_b^1! n_a^1! n_b^1! \ldots n_b^N!}{n_a^2! \ldots n_a^N! \bar{n}_b^2! \ldots \bar{n}_b^N!}} \\ &\sum_q \frac{(-1)^q (q+r)!(n_a + n_b + N - 2 - 2r - q)!}{q!(n_a - q - r)!(\bar{n}_b^1 - q - r)!(n_a^1 - n_a + q + r)!} \end{aligned} \tag{B.76}$$

Like in $SU(2)$ case, the denominator of (B.72) is the square-root of the numerator with $n_a^1 = n_a$, $n_a^2 = n_a^3 = \cdots = n_a^N = 0$ and $n_b^i = \bar{n}_b^i$, $\forall i$. The final expression for the denominator in (B.72) is:

$$\begin{aligned} & \Big\langle \{n_a^1 = n_a\}, \{\bar{n}_b\} \Big| \mathcal{P}_r \Big| \{n_a^1 = n_a\}, \{\bar{n}_b\} \Big\rangle \\ = \;& \frac{(n_a + n_b + N - 2r - 1) n_a! \bar{n}_b^1!}{r!(n_a + n_b + N - r - 1)!} \sum_q \frac{(-1)^q (n_a + n_b + N - 2 - 2r - q)!}{q!(n_a - q - r)!(\bar{n}_b^1 - q - r)!} \\ = \;& \frac{(n_a + n_b + N - 2r - 1) n_a! \bar{n}_b^1! (n_b + N - r - 2)!(n_a - \bar{n}_b^1 + n_b + N - 2 - r)!}{r!(n_a + n_b + N - r - 1)!(n_a - r)!(\bar{n}_b^1 - r)!(n_b - \bar{n}_b^1 + N - 2)!}. \end{aligned} \tag{B.77}$$

In (B.77) the last sum has been performed using (B.45) again. Finally, the $SU(N)$ Clebsch Gordon coefficient expansion (B.78) is obtained by dividing (B.76) with square root of

(B.77) as,

$$C_{\{n_a\},\{n_b\}}^{\;\;r} = \sqrt{\frac{(n_a+n_b+N-2r-1)n_a^1!n_b^1!n_b^2!...n_b^N!}{r!(n_a+n_b+N-r-1)!n_a^2!n_a^3!...n_a^N!\bar{n}_b^2!...\bar{n}_b^N!}}$$
$$\sqrt{\frac{(n_a-r)!(\bar{n}_b^1-r)!(n_b-\bar{n}_b^1+N-2)!}{(n_b+N-r-2)!(n_a+n_b-\bar{n}_b^1+N-r-2)!}}$$
$$\sum_q^{\min(n_a-r,n_b-r)} \frac{(-1)^q}{q!}\frac{(q+r)!(n_a+n_b+N-2-2r-q)!}{(n_a-q-r)!(\bar{n}_b^1-q-r)!(n_a^1-n_a+q+r)!} \quad \text{(B.78)}$$

Note that this $SU(N)$ Clebsh Gordon series reduces to the $SU(2)$ Clebsch Gordon series (B.47) for $N = 2$. This can be checked by identifying $(b_2^\dagger, b_1^\dagger)$ of $SU(N)$ with $(b_1^\dagger, -b_2^\dagger)$ of $SU(2)$ respectively so that $a^\dagger \cdot b^\dagger$ ($SU(N)$ invariant) $\rightarrow a^\dagger \cdot \tilde{b}^\dagger$ ($SU(2)$ invariant) and putting:

$$n_a^1 = j_1 + m_1 \qquad n_b^1 = j_2 - m_2 \qquad \bar{n}_b^1 = j_2 - (m - j_1)$$

$$n_a^2 = j_1 - m_1 \qquad n_b^2 = j_2 + m_2 \qquad \bar{n}_b^2 = j_2 + (m - j_1).$$

The $SU(N)$ Casimirs in (B.78) are: $n_a = n_a^1 + n_a^2 = 2j_1$, $\;n_b = n_b^1 + n_b^2 = \bar{n}_b^1 + \bar{n}_b^2 = 2j_2$ and $r = j_1 + j_2 - j$.

Bibliography

[1] K. Wilson, Phys. Rev. D10 (1974) 2445.

[2] J. Kogut, L. Susskind, Phys. Rev. D11 (1975) 395.

[3] M. Creutz, Quarks, Gluons and Lattices, Cambridge University Press, 1985.

[4] Heinz J. Rothe, Lattice Gauge Theories: An Introduction (Third Edition), World Scientific, 2005, John B. Kogut, Mikhail A. Stephanov, The Phases of Quantum Chromodynamics: ¿From Confinement to Extreme Environments, Cambridge University Press, 2003.

[5] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).
K. G. Wilson, Phys. Rept. **23**, 331 (1976).
T. L. Bell and K. G. Wilson, Phys. Rev. B **10**, 3935 (1974).
P. H. Ginsparg and K. G. Wilson, Phys. Rev. D **25**, 2649 (1982).
S. D. Glazek and K. G. Wilson, Phys. Rev. D **48**, 5863 (1993).
S. D. Glazek and K. G. Wilson, Phys. Rev. D **49**, 4214 (1994).
K. G. Wilson, Rev. Mod. Phys. **55**, 583 (1983).
K. G. Wilson, T. S. Walhout, A. Harindranath, W. -M. Zhang, R. J. Perry and S. D. Glazek, Phys. Rev. D **49**, 6720 (1994) [hep-th/9401153].

[6] J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).

[7] R. Gupta, NATO Adv. Study Inst. Ser. B Phys. **140**, 37 (1985), K. C. Bowler, R. D. Kenway, G. S. Pawley, D. J. Wallace, A. Hasenfratz, P. Hasenfratz, U. M. Heller and F. Karsch *et al.*, Nucl. Phys. B **257**, 155 (1985), A. S. Kronfeld, Ann. Rev. Nucl. Part. Sci. **62**, 265 (2012) [arXiv:1203.1204 [hep-lat]].

[8] K. G. Wilson and J. B. Kogut, Phys. Rept. **12**, 75 (1974).

[9] T. Banks *et al.*, Phys. Rev. D **15**, 1111 (1977).
J. Kogut, R. Pearson and J. Shigemitsu, Phys. Rev. Lett. 43 (1979) 484.
J. P. Greensite, Nucl. Phys. B **166**, 113 (1980).

[10] A. Patkos and F. Deak, Z. Phys. C **9**, 359 (1981).
N. D. Hari Dass, P. G. Lauwers and A. Patkos, Phys. Lett. B **124**, 387 (1983).
X. - Q. Luo, Q. - Z. Chen and S. - H. Guo, Z. Phys. C **47**, 635 (1990).
G. M. Frichter and D. Robson, Phys. Rev. D **52**, 320 (1995) [hep-lat/9312060].
H. Reinhardt, D. R. Campagnari, J. Heffner and M. Pak, Prog. Part. Nucl. Phys. **67**, 173 (2012) and the references therein.

[11] D. Horn and M. Weinstein, Phys. Rev. D **30**, 1256 (1984).

[12] L. C. L. Hollenberg, Phys. Rev. D **50**, 6917 (1994).

[13] LlewellynSmith:1992ig C. H. Llewellyn Smith and N. J. Watson, Phys. Lett. B **302**, 463 (1993) [hep-lat/9212025].

[14] D. Schutte, W. -h. Zheng and C. J. Hamer, Phys. Rev. D **55**, 2974 (1997) [hep-lat/9603026].
R. F. Bishop, N. Ligterink and N. R. Walet, Int. J. Mod. Phys. B **20**, 4992 (2006).
N. E. Ligterink, N. R. Walet and R. F. Bishop, Nucl. Phys. A **663**, 983 (2000).

[15] D. W. Heys and D. R. Stump, Phys. Rev. D **30**, 1315 (1984).
T. M. R. Byrnes, M. Loan, C. J. Hamer, F. D. R. Bonnet, D. B. Leinweber, A. G. Williams and J. M. Zanotti, Phys. Rev. D **69**, 074509 (2004) [hep-lat/0311014].

[16] J. Carlsson, hep-lat/0309138.

[17] D. Robson and D. M. Webber, Z. Phys. C **15**, 199 (1982).

[18] W. Furmanski and A. Kolawa, Nucl. Phys. B **291**, 594 (1987).

[19] J. S. Prakash and H. S. Sharatchandra 1996 *J. Math. Phys.* **37** 6530, hep-th/9607101;
J. S. Prakash and H. S. Sharatchandra 1993 *J. Phys. A* **26** 1625.

[20] Brügmann B, Phys. Rev. **D 43** (1991) 566.

[21] R. Gambini, Jorge Pullin, Loops, Knots, Gauge Theories and Quantum Gravity (Cambridge University Press, 2000).

[22] Gambini R, Leal L, Trias A, Phys. Rev. **D 39** (1989) 3127;
Bartolo C, Gambini R, Leal L, Phys. Rev. **D 39** (1989) 1756.

[23] Watson N. J., Phys. Letts. **B 323** (1994) 385;
Nucl. Phys. Proc. Suppl. (1995) 39BC 224, hep-th/9408174.

[24] J. Greensite, Nucl. Phys. **B 166** (1980) 113;
C. Hamer, A. Irving, T. Preece, Nucl. Phys. **B 270** (1986) 536;
Schütte D, Weihong Z, Hamer C J, Phys. Rev. **D 55** (1997) 2974;
S. Guo et. al., Phys. Rev. **D 49** (1994) 507.

[25] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **21**, R53 (2004) [gr-qc/0404018].

[26] C. Rovelli, "Quantum gravity," Cambridge, UK: Univ. Pr. (2004) 455 p

[27] T. Thiemann, "Modern canonical quantum general relativity," Cambridge, UK: Cambridge Univ. Pr. (2007) 819 p [gr-qc/0110034].

[28] Loll R. Nucl. Phys **B 368** (1992) 121, Nucl. Phys. **B 400** (1993) 126.

[29] Manu Mathur, Nucl. Phys. B **779**, 32 (2007);
Manu Mathur, Phys. Letts. **B 640** (2006) 292-296.

[30] Manu Mathur, J. Phys. A: Math. Gen. **38** (2005) 10015.

[31] S. Mandelstam, Ann. Phys. (N.Y.) **19** (1962) 1;
S. Mandelstam, Phys. Rev. **175** (1968) 1580;
S. Mandelstam, Phys. Rev. **D 19** (1979) 2391.

[32] R. Giles, Phys. Rev. **D 24** (1981) 2160.

[33] J. Schwinger U.S Atomic Energy Commission Report NYO-3071, 1952 or D. Mattis, *The Theory of Magnetism* (Harper and Row, 1982).

[34] R. Anishetty, M. Mathur and I. Raychowdhury, J. Math. Phys. **50**, 053503 (2009) [arXiv:0901.0644 [math-ph]].

[35] M. Mathur, I. Raychowdhury and R. Anishetty, J. Math. Phys. **51**, 093504 (2010) [arXiv:1003.5487 [math-ph]].

[36] R. Anishetty, M. Mathur and I. Raychowdhury, J. Phys. A **43**, 035403 (2010) [arXiv:0909.2394 [hep-lat]].

[37] A. A. Migdal, Phys. Rep. **102** 199-290 (1983).

[38] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific 1988).

[39] A. P. Yutsis, I. B. Levinson and V. V. Vanagas, *Mathematical Apparatus of the Theory of Angular Momentum* (Israel rogram for Scientific Translations, Jerusalem 1962).

[40] S. Chaturvedi and N. Mukunda, J. Math. Phys. **43** (2002) 5262.

[41] J. J. De Swart, Rev. Mod. Phys. **35**, 916 (1963).
J. D. Louck, Am. J. Phys. **38**, 3 (1970).
C. Itzykson, Rev. Mod. Phys. **38**, 95 (1966).
Arisaka N, Prog. Theor. Phys. **47**, 1758 (1972).
N. Mukunda and L. K. Pandit, J. Math. Phys. **6**, 746 (1965).
M. Resnikoff, J. Math. Phys. **8**, 63 (1967).
P. Jasselette, Nucl. Phys. B **1**, 521 (1967); ibid 529.
P. Jasselette, J. Phys. A: Math. Gen., **13**, 2261, (1980).
Biedenhm L C, J. Math. Phys. **4** 436 (1963).
R. Anishetty, H. Gopalkrishna Gadiyar, M. Mathur and H. S. Sharatchandra, Phys. Lett. B **271**, 391 (1991).

[42] M. Mathur and I. Raychowdhury, J. Phys. A **44**, 035203 (2011) [arXiv:1007.1510 [math-ph]].

[43] M. Mathur, I. Raychowdhury and T P Sreeraj, J. Math. Phys. **52**, 113505 (2011) [arXiv: 1108.5246 [math-ph]].

[44] H. S. Sharatchandra, Nucl. Phys. **B 196** (1982) 62.

[45] H. Georgi, Lie Algebras in Particle Physics (Benjamin/Cummings, Reading, 1982).

[46] M. Moshinsky, J. Math. Phys. **4**, 1128 (1963).

[47] D. Robson and D. M. Webber, Z. Phys. C **7**, 53 (1980). W. Furmanski, A. Kolawa, Nucl. Phys. **B 291** (1987) 594.

[48] W K Tung, Group Theory in Physics: An Introduction, World Scientific, 1985.

[49] Milton Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions, Courier Dover Publications, 1965.

[50] Belal E Baaquie J. Phys. A. **21** (1988) 2651.

[51] S. R. Wadia, Phys. Lett. B **93**, 403 (1980).

[52] O. L. Trinhammer and G. Olafsson, arXiv:math-ph/9901002.

[53] J. B. Kogut and J. Shigemitsu, Phys. Rev. Lett. **45**, 410 (1980) [Erratum-ibid. **45**, 1217 (1980)].

[54] M. Moshinsky, Rev. Mod. Phys. **34**, 813 1962; Bemstein I N, Gelfand I M and Gelfand S I 1976 Proc Pefrovskij Sem. 23, Reprinted in Gelfand I 1988 Collecfed Works VoL II (Berlin: Springer) p 464.

[55] Y. .Makeenko, Cambridge, UK: Univ. Pr. (2002) 417 p.
Yu. M. Makeenko and A. A. Migdal, Phys. Lett. **B88** (1979) 135;
Y. M. Makeenko, A. A. Migdal, Nucl. Phys. **B 188** (1981) 269.

[56] J. S. Prakash, hep-th/9604029.

[57] Ramesh Anishetty, H. S. Sharatchandra, Phys. Rev. Letts. **65** (1990) 81;
B. Gnanapragasam, H. S. Sharatchandra, Phys. Rev. **D 45** (1992) R1010.

[58] G. Burgio, R. De. Pietri, H. A. Morales-Tecotl, L. F. Urrutia, J. D. Vergara, Nucl. Phys. **B 566** (2000), 547.

[59] Manu Mathur and Diptiman Sen, J. Math. Phys. **42** (2001) 4181.

[60] Manu Mathur and H. S. Mani, J. Math. Phys. **43** (2002) 5351.

[61] E. Schrödinger, Naturwissenshafter, **14**, 644, (1926).

[62] Klauder J R and Skagerstam B-S 1985 Coherent States: Applications in Physics and Mathematical Physics (Singapore: World Scientic),
Inomata A, Kuratsuji H and Gerry C C 1992 Path Integrals and Coherent States of SU (2) and SU (1, 1) (Singapore: World Scientic).

[63] R. J. Glauber, Phys. Rev. **130** (1963) 2529.

[64] E. C. G. Sudarshan, Phys. Rev. Lett. **10** 277;
E. C. G. Sudarshan, Phys. Rev. **131** (1963) 2766.

[65] W. M. Zhang, D. H. Feng and R. Gilmore, "Coherent states: Theory and some Applications," Rev. Mod. Phys. **62**, 867 (1990).

[66] Perelomov A 1986 Generalized Coherent States and Their Applications (New York: Springer),
Perelomov A 1977 Sov. Phys.Usp. 20 703,
A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).

[67] A. Auerbach, Interacting Electrons and Quantum Magnetism , (Springer-Verlag, N.Y.), A. Auerbach and D.P. Arovas, in Field Theories In Condensed Matter Physics, Ed. Z. Tesanovich, Addison-Wesley (1990).

[68] A. Klein E. R. Marshalek, Rev. Mod. Phys. 63, 375-557 (1991).